

ENTIRE SPACELIKE TRANSLATING SOLITONS IN MINKOWSKI SPACE

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ABSTRACT. In this paper, we study entire spacelike translating solitons in Minkowski space. By constructing convex spacelike solutions to (1.3) in bounded convex domains, we obtain many entire smooth convex strictly spacelike translating solitons by prescribing boundary data at infinity.

1. INTRODUCTION

Let \mathbb{R}_1^{n+1} be the Minkowski space $(\mathbb{R}^{n+1}, \bar{g})$ with the Lorentz metric

$$\bar{g} = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2.$$

We will say that a hypersurface $\Sigma = \{(x, w(x)) \mid x \in \Omega\} \subset \mathbb{R}_1^{n+1}$ is *strictly spacelike*, if $w \in C^1(\Omega)$ and $|Dw| < 1$ in Ω ; Σ is *weakly spacelike*, if $w \in C^{0,1}(\Omega)$ and $|Dw| \leq 1$ a.e. in Ω . Here, $C^{0,1}(\Omega)$ is the class of locally Lipschitz functions in Ω . For convenience, we often call strictly (weakly) spacelike hypersurfaces by the functions whose graphs are them.

Mean curvature flow in Minkowski space is a family of smooth strictly spacelike embeddings $X_t = X(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{R}_1^{n+1}$ with corresponding images $M_t = X_t(\mathbb{R}^n)$ satisfying the following evolution equation

$$(1.1) \quad \frac{dX}{dt} = \vec{H}$$

on some time interval, where \vec{H} is the mean curvature vector of M_t in \mathbb{R}_1^{n+1} . Every M_t is the graph of a function $U(\cdot, t)$ with $|DU(\cdot, t)| < 1$. Equation (1.1) is equivalent up to diffeomorphisms in \mathbb{R}^n to the equation

$$(1.2) \quad \frac{\partial U}{\partial t} = \sqrt{1 - |DU|^2} \operatorname{div} \left(\frac{DU}{\sqrt{1 - |DU|^2}} \right),$$

where 'div' is the divergence on \mathbb{R}^n . There is an important class of solutions for (1.2) in \mathbb{R}^n which moves by vertical translation. This solution is called *Translating Soliton*, i.e.,

$$x_{n+1} = U(x, t) = u(x) + t,$$

where u satisfies the elliptic equation

$$(1.3) \quad \operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = \frac{1}{\sqrt{1 - |Du|^2}} \quad \text{in } \mathbb{R}^n.$$

Mean curvature flow in the ambient Minkowski space and Lorentzian manifold has been studied extensively (see [7][8][9][10][15] for example). Translating solitons can be regarded as a natural way of foliating spacetimes by almost null like hypersurfaces. Particular examples may give insight into the structure of certain spacetimes at null infinity and have possible applications in general relativity [8]. Convex translating solitons in Euclidean space \mathbb{R}^{n+1} , namely, the graphic functions satisfy

$$(1.4) \quad \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{1}{\sqrt{1 + |Du|^2}} \quad \text{in } \mathbb{R}^n,$$

are arising at type II singularities of mean curvature flow by Huisken and Sinestrari [13][14]. X.-J. Wang in [20] showed that the convex solutions to (1.4) must be rotationally symmetric in an appropriate coordinate system for $n = 2$, and constructed non-rotationally symmetric entire convex translating solitons for $n \geq 3$.

In the case $n = 1$, $u_0(x) = \log \cosh x$ is a particular solution to (1.3). In [8], Ecker constructed a radially symmetric solution to (1.3) for general n . Later, H.-Y. Jian [16] gave a detailed discussion for this radially symmetric solution. In this paper, we only consider the case $n \geq 2$ and construct many entire smooth convex strictly spacelike translating solutions which are asymptotic to all the functions in Q but linear functions at infinity.

Here Q is a set defined as follows: if $w \in Q$, then w is a convex homogeneous of degree one function and $|Dw(y)| = 1$ for any $y \in \mathbb{R}^n$ where the gradient exists. Each function $w \in Q$ could be represented as $w(x) = \sup_{\lambda \in \Lambda} \langle \lambda, x \rangle$ where $\Lambda \subset \mathbb{S}^{n-1}$ is a closed set of unit vectors and $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{R}^n . Conversely, for every closed set $\Lambda \subset \mathbb{S}^{n-1}$, $\sup_{\lambda \in \Lambda} \langle \lambda, x \rangle \in Q$ (please see [4][19] for details). For any weakly spacelike function w in \mathbb{R}^n , we define

$$V_w(x) = \lim_{r \rightarrow \infty} \frac{w(rx)}{r}$$

if such a limit exists for every $x \in \mathbb{R}^n$. And we say that V_w is the *blow down* of w . The blow down is well-defined for convex weakly spacelike functions [19].

A famous Calabi-Cheng-Yau [3][6] result tells us non-existence of nontrivial complete maximal spacelike hypersurface in Minkowski space. Cheng-Yau [6] also showed that any spacelike hypersurface Σ of nonzero constant mean curvature has nonpositive Ricci

curvature, namely that the function whose graph is Σ is convex (see also [19]). For every function $V \in Q$ but linear functions Treibergs in [19] could find a function u blowing down to V and the graph of u is a strictly spacelike constant mean curvature hypersurface. Compared to this work of Treibergs we get the following Theorem for translating solitons.

Theorem 1.1. *Let Q be defined as above. For $n \geq 2$ and any function V in Q except linear functions there is an entire smooth convex strictly spacelike solution u to (1.3) such that u blows down to V , namely, $\lim_{r \rightarrow \infty} \frac{u(rx)}{r} = V(x)$ for each $x \in \mathbb{R}^n$.*

In [1], M. Aarons gave a conjecture which asks: *For $a > 0$, whether there is a solution u of*

$$(1.5) \quad \operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = \frac{a}{\sqrt{1 - |Du|^2}} + c \quad \text{in } \mathbb{R}^n,$$

which blows down to every V in Q .

We may use the technique for proving Theorem 1.1 to study (1.5) with $c > 0$, and obtain an existence result for (1.5) in § 6.

Every entire spacelike graph with constant mean curvature hypersurface in \mathbb{R}_1^{n+1} is complete [6]. However, translating solitons in \mathbb{R}_1^{n+1} may be not complete, and the mean curvatures must be unbounded. A nature question is whether every entire strictly spacelike function u to (1.3) is convex? H.-Y. Jian also asked this question in [16].

In the present paper, we find a variational functional F defined by (2.1) for equation (1.3), and show that any translating soliton is maximal for F , namely, any variation with compact support do not increase its area under the weight $e^{-x_{n+1}}$ (see Theorem 2.1). This help us to establish a comparison principle of weak solutions to F . By constructing barrier functions we study light ray within the weak solution hypersurface, which plays a key role for showing that the limit function of a sequence of strictly spacelike solutions to (1.3) is strictly spacelike.

Through a calculation for the second fundamental form of translating solitons, convexity of the bounded level set for any solution to (1.3) could imply convexity of the corresponding sublevel set with a restriction of the mean curvature of the level set. Then it is able to solve a class of Dirichlet problems in smooth bounded convex domains, see Theorem 4.6. We construct auxiliary functions to seek out a sequence of convex solutions $\{u_k\}$ to (1.3) in different bounded domains which is asymptotic to prescribing function at infinity.

Convexity of u_k could enable us to get the uniform bound for Hessian of u_k , which is crucial to show that their limit u is the desired function in Theorem 1.1.

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2. TRANSLATING SOLITONS FROM VARIATIONAL VIEW

In this section, we always suppose that Ω is a bounded domain in \mathbb{R}^n . Let $M = \{(x, u(x)) \mid x \in \Omega\}$ be a weakly spacelike hypersurface in \mathbb{R}_1^{n+1} with volume element $d\mu = \sqrt{1 - |Du|^2} dx$. We define a functional F by

$$(2.1) \quad F(M) = \int_M e^{-x_{n+1}} d\mu = \int_\Omega e^{-u(x)} \sqrt{1 - |Du|^2} dx.$$

If u is smooth and $|Du| < 1$ in Ω , then M is a Riemannian manifold with the metric $g_{ij} dx_i dx_j$, where $g_{ij} = \delta_{ij} - u_i u_j$. Let $\langle \cdot, \cdot \rangle$ be indefinite inner product in \mathbb{R}_1^{n+1} endowed by the Lorentz metric \bar{g} (This definition does not conflict with n -dimensional Euclidean inner product $\langle \cdot, \cdot \rangle$ in § 1). The normal vector field $\nu = \frac{1}{\sqrt{1 - |Du|^2}} (\sum_{i=1}^n u_i E_i + E_{n+1})$ satisfies $\langle \nu, \nu \rangle = -1$, where E_1, \dots, E_{n+1} is the unit natural basis of \mathbb{R}_1^{n+1} .

If M_s is a variation of M with $s \in (-1, 1)$ and $M_0 = M$, where the variation vector field $M'_0 = f\nu$ and f is smooth with $f|_{\partial M_0} \equiv 0$. By a calculation (see [21] for Euclidean case or (2.16) for $W \equiv 0$), we have

$$(2.2) \quad \left. \frac{d}{ds} F(M_s) \right|_{s=0} = \int_M (-fH - f\langle \nu, E_{n+1} \rangle) e^{-x_{n+1}} d\mu,$$

where $H = \operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right)$. We will say that M is a *critical point* for the functional F if it is critical with respect to all normal variations in M . Hence, M is a critical point for F if and only if the mean curvature of M satisfies

$$(2.3) \quad H = -\langle \nu, E_{n+1} \rangle.$$

(2.3) is just (1.3). Let $g^{ij} = \delta_{ij} + \frac{u_i u_j}{1 - |Du|^2}$, then (1.3) can be rewritten as

$$(2.4) \quad \sum_{i,j} g^{ij} u_{ij} = \sum_{i,j} \left(\delta_{ij} + \frac{u_i u_j}{1 - |Du|^2} \right) u_{ij} = 1.$$

Let φ be a weakly spacelike function on $\bar{\Omega}$ and $\mathcal{C}(\varphi, \Omega)$ be a set defined by

$$\{w \in C^{0,1}(\Omega) \mid w = \varphi \text{ on } \partial\Omega, \text{ and } |Dw| \leq 1 \text{ a.e. in } \Omega\}.$$

For any $\epsilon > 0$, denote $\Omega_\epsilon = \{x \in \Omega \mid \inf_{y \in \partial\Omega} |x-y| \geq \epsilon\}$ and $\Omega^\epsilon = \{x \in \mathbb{R}^n \mid \inf_{y \in \overline{\Omega}} |x-y| \leq \epsilon\}$. By the boundedness of $\overline{\Omega}$, Ω_ϵ and Ω^ϵ are both closed sets. Let ρ be a smooth function with compact support in $B_1(0) \subset \mathbb{R}^n$ and $\int_{\mathbb{R}^n} \rho(x) dx = 1$. Let w_ϵ be a mollifier of weakly spacelike function $w \in \mathcal{C}(\varphi, \Omega)$ defined by

$$(2.5) \quad w_\epsilon(x) = (w * \rho_\epsilon)(x) \triangleq \int_{\mathbb{R}^n} \rho(y) w(x - \epsilon y) dy \quad \text{for } x \in \Omega_\epsilon.$$

Then

$$(2.6) \quad \begin{aligned} |Dw_\epsilon(x)|^2 &= \sum_i \left| \int_{\mathbb{R}^n} \rho(y) \partial_{x_i} w(x - \epsilon y) dy \right|^2 \leq \sum_i \int_{\mathbb{R}^n} \rho(y) dy \int_{\mathbb{R}^n} \rho(y) |\partial_{x_i} w(x - \epsilon y)|^2 dy \\ &= \int_{\mathbb{R}^n} \rho(y) |Dw(x - \epsilon y)|^2 dy \leq \int_{\mathbb{R}^n} \rho(y) dy = 1. \end{aligned}$$

Moreover, $w_\epsilon \rightarrow w$ uniformly and $Dw_\epsilon \rightarrow Dw$ a.e. in any compact set $K \subset \Omega$.

Theorem 2.1. *Let Ω be a bounded domain in \mathbb{R}^n and u be a smooth strictly spacelike solution to (1.3) in $\overline{\Omega}$. Set $M = \{(x, u(x)) \mid x \in \Omega\}$, then for any bounded weakly spacelike hypersurface $\Sigma \subset \mathbb{R}_1^{n+1}$ with $\partial\Sigma = \partial M$, one has*

$$(2.7) \quad \int_{\Sigma} e^{-x_{n+1}} dV_{\Sigma} \leq \int_M e^{-x_{n+1}} dV_M,$$

where the above inequality attains equality if and only if $\Sigma = M$.

Proof. There is a domain $\tilde{\Omega}$ with $\Omega \subset \subset \tilde{\Omega}$ such that u can be extended to $\tilde{\Omega}$ smoothly, and we may assume $|Du| < 1$ in $\tilde{\Omega}$. Σ can be written as $\{(x, w(x)) \mid x \in \Omega\}$ for some $w \in \mathcal{C}(u, \Omega)$. Since $w|_{\partial\Omega} = u|_{\partial\Omega}$, then we extend w to $\tilde{\Omega}$ with $w|_{\tilde{\Omega} \setminus \Omega} = u|_{\tilde{\Omega} \setminus \Omega}$. Clearly, there is a $\epsilon_0 > 0$ such that $\Omega^{\epsilon_0} \subset \tilde{\Omega}$. Since u is smooth in Ω^{ϵ_0} , then there exists a constant C depending only on n and $\sup_{\Omega^{\epsilon_0}} |D^2 u|$, such that for any $0 < \epsilon \leq \frac{\epsilon_0}{2}$ and $x \in \Omega^\epsilon$ one has

$$(2.8) \quad |D(u * \rho_\epsilon)(x) - Du(x)| = \left(\sum_i \left| \int_{\mathbb{R}^n} \rho(y) (\partial_{x_i} u(x - \epsilon y) - \partial_{x_i} u(x)) dy \right|^2 \right)^{\frac{1}{2}} \leq C\epsilon.$$

Set $\tilde{w}_\epsilon = (w - u) * \rho_\epsilon$ and $w_\epsilon = (1 - C\epsilon)(\tilde{w}_\epsilon + u)$. Then w_ϵ is a smooth function in $\Omega^{\frac{\epsilon_0}{2}}$ and $w_\epsilon(x) = (1 - C\epsilon)u(x)$ for any $x \in \partial\Omega^\epsilon$ and $0 < \epsilon \leq \frac{\epsilon_0}{2}$. Moreover $w_\epsilon \rightarrow w$ uniformly and $Dw_\epsilon \rightarrow Dw$ a.e. in $\overline{\Omega}$. By (2.8), in Ω^ϵ one obtains

$$(2.9) \quad \begin{aligned} |Dw_\epsilon| &= (1 - C\epsilon) |D(w * \rho_\epsilon - u * \rho_\epsilon + u)| \\ &\leq (1 - C\epsilon) |D(w * \rho_\epsilon)| + (1 - C\epsilon) |D(u * \rho_\epsilon - u)| \\ &\leq 1 - C\epsilon + C\epsilon(1 - C\epsilon) = 1 - C^2\epsilon^2. \end{aligned}$$

Let $M_\epsilon = \{(x, (1 - C\epsilon)u(x)) \mid x \in \Omega^\epsilon\}$ and $\Sigma_\epsilon = \{(x, w_\epsilon(x)) \mid x \in \Omega^\epsilon\}$, then they are both smooth strictly spacelike hypersurfaces with $\partial M_\epsilon = \partial \Sigma_\epsilon$. Let D_ϵ be the domain enclosed by M_ϵ and Σ_ϵ . Let $v = \sqrt{1 - |Du|^2}$ and Y be a vector field in M_ϵ defined by

$$Y = \sum_{i=1}^n \frac{u_i}{v} e^{-x_{n+1}} E_i + \frac{e^{-x_{n+1}}}{v} E_{n+1}.$$

Viewing u_i and v as functions of x_1, \dots, x_n and translating Y to Σ_ϵ along the x_{n+1} axis. Then we obtain a vector field in D_ϵ , denoted by Y , too. Let $\bar{\nabla}$ and $\bar{\text{div}}$ be Levi-Civita connection and divergence on \mathbb{R}_1^{n+1} with the Lorentz metric \bar{g} , respectively. From [11], one has

$$\begin{aligned} \bar{\text{div}}(Y) &= \sum_i \langle \bar{\nabla}_{E_i} Y, E_i \rangle - \langle \bar{\nabla}_{E_{n+1}} Y, E_{n+1} \rangle \\ &= \sum_i \langle E_i, [E_i, Y] \rangle - \langle E_{n+1}, [E_{n+1}, Y] \rangle \\ (2.10) \quad &= \sum_i \partial_{x_i} \left(\frac{u_i}{v} e^{-x_{n+1}} \right) + \partial_{x_{n+1}} \left(\frac{e^{-x_{n+1}}}{v} \right) \\ &= \sum_i \partial_{x_i} \left(\frac{u_i}{v} \right) e^{-x_{n+1}} - \frac{1}{v} e^{-x_{n+1}}. \end{aligned}$$

Let $\nu_\epsilon, \nu_{\Sigma_\epsilon}$ be the timelike future-pointing unit normal vectors of $M_\epsilon, \Sigma_\epsilon$ respectively, then by Gauss formula (see [11] for example), up to a minus sign we have

$$(2.11) \quad \int_{D_\epsilon} \bar{\text{div}}(Y) = \int_{M_\epsilon} \langle Y, \nu_\epsilon \rangle dV_{M_\epsilon} - \int_{\Sigma_\epsilon} \langle Y, \nu_{\Sigma_\epsilon} \rangle dV_{\Sigma_\epsilon}.$$

In fact, let the orientation of ∂D_ϵ direct timelike future-pointing in $M_\epsilon \setminus \Sigma_\epsilon$ and direct timelike past-pointing in $\Sigma_\epsilon \setminus M_\epsilon$. Let $(\gamma_1, \dots, \gamma_{n+1})$ with $\gamma_{n+1}^2 - \sum_i \gamma_i^2 = 1$ be the timelike unit normal vector of $\partial D_\epsilon \setminus (M_\epsilon \cap \Sigma_\epsilon)$ with respect to the orientation of ∂D_ϵ . Set $(\gamma_1, \dots, \gamma_{n+1})$ be an arbitrary constant vector in $M_\epsilon \cap \Sigma_\epsilon$, then we have

$$\begin{aligned} \int_{D_\epsilon} \bar{\text{div}}(Y) &= - \int_{D_\epsilon} \left(\sum_i \partial_{x_i} \left(\frac{u_i}{v} e^{-x_{n+1}} \right) + \partial_{x_{n+1}} \left(\frac{e^{-x_{n+1}}}{v} \right) \right) dx_1 \wedge \dots \wedge dx_{n+1} \\ &= \sum_i (-1)^i \int_{\partial D_\epsilon} \frac{u_i}{v} e^{-x_{n+1}} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_{n+1} + (-1)^{n+1} \int_{\partial D_\epsilon} \frac{e^{-x_{n+1}}}{v} dx_1 \wedge \dots \wedge dx_n \\ &= \sum_i \int_{\partial D_\epsilon} \frac{u_i}{v} e^{-x_{n+1}} \gamma_i dV_{\partial D_\epsilon} - \int_{\partial D_\epsilon} \frac{e^{-x_{n+1}}}{v} \gamma_{n+1} dV_{\partial D_\epsilon} \\ &= \int_{M_\epsilon} \langle Y, \nu_\epsilon \rangle dV_{M_\epsilon} + \int_{\Sigma_\epsilon} \langle Y, -\nu_{\Sigma_\epsilon} \rangle dV_{\Sigma_\epsilon}. \end{aligned}$$

If ξ and η are timelike future-pointing vectors in \mathbb{R}_1^{n+1} , then reversed Cauchy-Schwarz inequality implies (see[18] for example)

$$-\langle \xi, \eta \rangle \geq \sqrt{|\langle \xi, \xi \rangle|} \sqrt{|\langle \eta, \eta \rangle|}.$$

Substituting it into (2.11) gives

$$(2.12) \quad \begin{aligned} \int_{D_\epsilon} \overline{\operatorname{div}}(Y) - \int_{M_\epsilon} \langle Y, \nu_\epsilon \rangle dV_{M_\epsilon} &= - \int_{\Sigma_\epsilon} \langle Y, \nu_{\Sigma_\epsilon} \rangle dV_{\Sigma_\epsilon} \\ &\geq \int_{\Sigma_\epsilon} \sqrt{|\langle Y, Y \rangle|} dV_{\Sigma_\epsilon} = \int_{\Sigma_\epsilon} e^{-x_{n+1}} dV_{\Sigma_\epsilon}. \end{aligned}$$

The inequality (2.12) arrives at equality if and only if Y parallels ν_{Σ_ϵ} . By (1.3), $\overline{\operatorname{div}}(Y) = 0$ in Ω . Since u is strictly spacelike, then $\int_{D_\epsilon} \overline{\operatorname{div}}(Y) \rightarrow 0$ as $\epsilon \rightarrow 0$. Combining $\langle Y, \nu_\epsilon \rangle \rightarrow -e^{-x_{n+1}}$ and (2.12), we obtain the desired result. \square

For any C^2 function w with $|Dw| < 1$, we define a differential operator by

$$\mathcal{L}w = \sum_{i,j} \left(\delta_{ij} + \frac{w_i w_j}{1 - |Dw|^2} \right) w_{ij}.$$

Lemma 2.2. *Let $u, \bar{u}, \underline{u}$ be three C^2 strictly spacelike functions satisfying $\mathcal{L}u = 1, \mathcal{L}\bar{u} \leq 1, \mathcal{L}\underline{u} \geq 1$ in Ω , then*

$$(2.13) \quad \begin{aligned} u(x) - \bar{u}(x) &\leq \sup_{y \in \partial\Omega} (u(y) - \bar{u}(y)) \quad \text{for } x \in \Omega, \\ u(x) - \underline{u}(x) &\geq \inf_{y \in \partial\Omega} (u(y) - \underline{u}(y)) \quad \text{for } x \in \Omega. \end{aligned}$$

Proof. Let $w = u - \bar{u}$, then

$$(2.14) \quad \begin{aligned} 0 \leq \mathcal{L}u - \mathcal{L}\bar{u} &= \left(\delta_{ij} + \frac{u_i u_j}{1 - |Du|^2} \right) w_{ij} + \left(\frac{u_i u_j}{1 - |Du|^2} - \frac{\bar{u}_i \bar{u}_j}{1 - |D\bar{u}|^2} \right) \bar{u}_{ij} \\ &= \left(\delta_{ij} + \frac{u_i u_j}{1 - |Du|^2} \right) w_{ij} + \frac{(|Du|^2 - |D\bar{u}|^2) u_i u_j + (1 - |Du|^2)(u_i u_j - \bar{u}_i \bar{u}_j)}{(1 - |Du|^2)(1 - |D\bar{u}|^2)} \bar{u}_{ij} \\ &= \left(\delta_{ij} + \frac{u_i u_j}{1 - |Du|^2} \right) w_{ij} + \frac{u_i u_j (u_k + \bar{u}_k) w_k + (1 - |Du|^2)(u_i u_j - \bar{u}_i \bar{u}_j)}{(1 - |Du|^2)(1 - |D\bar{u}|^2)} \bar{u}_{ij}. \end{aligned}$$

By the maximum principle of elliptic equations, we have $w(x) \leq \sup_{y \in \partial\Omega} w(y)$ for each $x \in \Omega$. Clearly, one could prove the second inequality in Lemma 2.2 by the same method. \square

Let W be a continuous function in Ω . We define a functional $F_{W,\Omega}$ on a function $w \in \mathcal{C}(\varphi, \Omega)$ by

$$F_{W,\Omega}(w) = \int_{\Omega} e^{-w} \left(\sqrt{1 - |Dw|^2} + W \right) dx.$$

Denote $F_{0,\Omega}(\cdot)$ by $F_\Omega(\cdot)$ for simplicity. The Dirichlet problem

$$(2.15) \quad \begin{cases} \operatorname{div} \left(\frac{Dw}{\sqrt{1-|Dw|^2}} \right) - \frac{1}{\sqrt{1-|Dw|^2}} = W & \text{in } \Omega, \\ w(x) = \varphi(x) & \text{for } x \in \partial\Omega, \end{cases}$$

raises from the Euler-Lagrange equation of the variational problem $\sup_{z \in \mathcal{C}(\varphi, \Omega)} F_{W,\Omega}(z)$.

In fact, for any $\eta \in C_c^\infty(\Omega)$ a simply calculation gives

$$(2.16) \quad \begin{aligned} \left. \frac{d}{dt} \right|_{t=0} F_{W,\Omega}(w + t\eta) &= \int_{\Omega} \left(-\eta e^{-w} \left(\sqrt{1-|Dw|^2} + W \right) - e^{-w} \frac{Dw \cdot D\eta}{\sqrt{1-|Dw|^2}} \right) \\ &= \int_{\Omega} \eta e^{-w} \left(\operatorname{div} \left(\frac{Dw}{\sqrt{1-|Dw|^2}} \right) - \frac{1}{\sqrt{1-|Dw|^2}} - W \right). \end{aligned}$$

Lemma 2.3. *Let u be a strictly spacelike function to (2.4) in Ω with $u|_{\partial\Omega} = \varphi$. Let φ_1, φ_2 be weakly spacelike functions in $\overline{\Omega}$ and W_1 be a nonpositive continuous function and W_2 be a nonnegative continuous function in Ω . If $w_i \in \mathcal{C}(\varphi_i, \Omega)$ and $F_{W_i,\Omega}(w_i) = \sup_{w \in \mathcal{C}(\varphi_i, \Omega)} F_{W_i,\Omega}(w)$ for $i = 1, 2$, then*

$$(2.17) \quad \begin{aligned} u(x) &\leq w_1(x) + \sup_{\partial\Omega} (\varphi - \varphi_1) & \text{for } x \in \Omega \\ u(x) &\geq w_2(x) + \inf_{\partial\Omega} (\varphi - \varphi_2) & \text{for } x \in \Omega. \end{aligned}$$

Proof. Let $C = \sup_{\partial\Omega} (\varphi - \varphi_1)$, ϵ be a small positive constant, $w^* = w_1 + C + \epsilon$ and $\Omega^+ = \{x \in \Omega \mid u(x) > w^*(x)\}$. Assume $\Omega^+ \neq \emptyset$.

Clearly, $\overline{\Omega^+} \subset \Omega$ and $u(x) = w^*(x)$ on $\partial\Omega^+$. By $F_{W_1,\Omega^+}(w^*) \geq F_{W_1,\Omega^+}(u)$, we conclude

$$(2.18) \quad \begin{aligned} \int_{\Omega^+} e^{-w^*} \sqrt{1-|Dw^*|^2} dx &= F_{W_1,\Omega^+}(w^*) - \int_{\Omega^+} e^{-w^*} W_1 dx \\ &\geq F_{W_1,\Omega^+}(u) - \int_{\Omega^+} e^{-w^*} W_1 dx \\ &= \int_{\Omega^+} e^{-u} \sqrt{1-|Du|^2} dx + \int_{\Omega^+} (e^{-u} - e^{-w^*}) W_1 dx \\ &\geq \int_{\Omega^+} e^{-u} \sqrt{1-|Du|^2} dx. \end{aligned}$$

Theorem 2.1 implies $F_{\Omega^+}(w^*) \leq F_{\Omega^+}(u)$, then $F_{\Omega^+}(w^*) = F_{\Omega^+}(u)$ and $W_1 \equiv 0$. Hence Ω^+ is empty by Theorem 2.1. Letting $\epsilon \rightarrow 0$ yields the first inequality in (2.17). The second inequality in (2.17) could be showed similarly. \square

Remark 2.4. *Lemma 2.3 can be seen as a weak version of Lemma 2.2, if we set $W_1 =$*

$$\frac{\mathcal{L}\overline{u}-1}{\sqrt{1-|D\overline{u}|^2}} \text{ and } W_2 = \frac{\mathcal{L}u-1}{\sqrt{1-|Du|^2}}.$$

3. BARRIER FUNCTIONS AND APPLICATIONS

For $K \in \mathbb{R}$, let

$$(3.1) \quad w_K(t) \triangleq \int_0^t \frac{K}{\sqrt{s^{2n-2} + K^2}} ds \quad \text{and} \quad \tilde{w}_K(t) \triangleq \int_0^t \frac{K}{\sqrt{s^{2n} + K^2}} ds \quad \text{for } t \geq 0.$$

Then

$$(3.2) \quad \mathcal{L}w_K = \frac{w_K''}{1 - (w_K')^2} + \frac{n-1}{t} w_K' = t^{1-n} \sqrt{1 - (w_K')^2} \left(\frac{t^{n-1} w_K'}{\sqrt{1 - (w_K')^2}} \right)' = 0,$$

and

$$(3.3) \quad \mathcal{L}\tilde{w}_K = \frac{\tilde{w}_K''}{1 - (\tilde{w}_K')^2} + \frac{n-1}{t} \tilde{w}_K' = -\frac{K}{t\sqrt{t^{2n} + K^2}}.$$

Moreover, $\lim_{K \rightarrow \pm\infty} w_K(t) = \lim_{K \rightarrow \pm\infty} \tilde{w}_K(t) = \pm t$.

Now we give an existence theorem for an ODE arising from (2.4).

Theorem 3.1. *For any constant $C \in (-r, r)$ and $r > 0$, the following ODE:*

$$(3.4) \quad \begin{cases} \frac{\phi''}{1 - (\phi')^2} + \frac{n-1}{t} \phi' = 1 & \text{for } t \in (0, r), \\ \phi(r) = C, \phi(0) = 0 \text{ and } |\phi'| < 1, \end{cases}$$

has a unique smooth solution ϕ_0 in $(0, r)$. Furthermore, if $r^2 \leq (n-1)C$ and $K_1 > 0$ is a constant with $w_{K_1}(r) = C$, then

$$\frac{C}{r} t \leq \phi_0(t) \leq w_{K_1}(t) \quad \text{for } t \in [0, r].$$

If $C < 0$, $r < 1$, $K_2 < 0$ is a constant with $\tilde{w}_{K_2}(r) = C$ and $K_2^2 \geq \frac{r^{2n+2}}{1-r^2}$, then

$$\tilde{w}_{K_2}(t) \leq \phi_0(t) \leq \frac{C}{r} t \quad \text{for } t \in [0, r].$$

Proof. We consider a family of approximation equations:

$$(3.5) \quad \begin{cases} \frac{\phi''}{1 - (\phi')^2} + \frac{n-1}{t} \phi' = 1 & \text{for } t \in [\epsilon, r), \\ \phi(r) = C, \phi(\epsilon) = 0 \text{ and } |\phi'| < 1. \end{cases}$$

Clearly, (3.5) has a smooth solution ϕ_ϵ on $[\epsilon, r)$ for $0 < \epsilon < r - |C|$. For any fixed $\delta \in (0, r)$ there is a sequence $\{\epsilon_i\}$ such that $\phi_{\epsilon_i} \rightarrow \phi_0$ uniformly in $[\delta, r)$ as $\epsilon_i \rightarrow 0$. Then $\phi_0 \in C^{0,1}((0, r))$, $\phi_0(r) = C$, $\phi_0(0) = 0$ and $|\phi_0'| \leq 1$ a.e..

If there is a subsequence $\epsilon_{i_l} \rightarrow 0$ (we denoted it by ϵ_i) such that $\phi'_{\epsilon_i}(t_0) \rightarrow \alpha$ for some fixed $t_0 \in (0, r)$, then integrating (3.5) for ϕ'_ϵ with sufficiently small $0 < \epsilon < t_0$ gives

$$(3.6) \quad \frac{1}{2} \log \left| \frac{1 + \phi'_\epsilon(t)}{1 - \phi'_\epsilon(t)} \right|_{t_0}^t + (n-1) \int_{t_0}^t \frac{\phi'_\epsilon(s)}{s} ds = t - t_0 \quad \text{for } t \in [\epsilon, r).$$

If $\alpha = \pm 1$, then letting $\epsilon_i \rightarrow 0^+$ gets $\phi'_{\epsilon_i}(t) \rightarrow \pm 1$ for any $t \in (0, r)$. Set

$$f_\epsilon(t) = \frac{1 + \phi'_\epsilon(t_0)}{1 - \phi'_\epsilon(t_0)} e^{2(t-t_0)-2(n-1) \int_{t_0}^t \frac{\phi'_\epsilon(s)}{s} ds},$$

then (3.6) implies $\frac{1+\phi'_\epsilon(t)}{1-\phi'_\epsilon(t)} = f_\epsilon(t)$ and

$$(3.7) \quad \phi_\epsilon(t) - \phi_\epsilon(t_0) = \int_{t_0}^t \frac{f_\epsilon(s) - 1}{f_\epsilon(s) + 1} ds.$$

Thus one has $\phi_0(t) - \phi_0(t_0) = \pm(t - t_0)$ which contradicts with $\phi_0(r) - \phi_0(0) = C$. Therefore, $\phi'_{\epsilon_i}(t_0) \rightarrow \alpha \in (-1, 1)$, then by (3.7) and $\phi_0 \in C^{0,1}((0, r))$, we obtain

$$(3.8) \quad \phi_0(t) - \phi_0(t_0) = \int_{t_0}^t \frac{\alpha - 1 + (1 + \alpha) e^{2(s-t_0)-2(n-1) \int_{t_0}^s \frac{\phi'_0(p)}{p} dp}}{1 - \alpha + (1 + \alpha) e^{2(s-t_0)-2(n-1) \int_{t_0}^s \frac{\phi'_0(p)}{p} dp}} ds \quad \text{for } t > 0.$$

Hence $\phi'_0(t)$ exists everywhere in $(0, r)$, and one gets

$$(3.9) \quad \frac{1}{2} \log \frac{1 + \phi'_0(t)}{1 - \phi'_0(t)} - \frac{1}{2} \log \frac{1 + \alpha}{1 - \alpha} + (n-1) \int_{t_0}^t \frac{\phi'_0(s)}{s} ds = t - t_0.$$

Then $\phi'_0(t)$ is continuous and $\phi''_0(t)$ exists everywhere in $(0, r)$. Now (3.9) implies

$$\phi''_0 = (1 - (\phi'_0)^2) \left(1 - \frac{n-1}{t} \phi'_0 \right).$$

The above equation shows that ϕ_0 is our desired smooth solution to (3.4). By Lemma 2.2, we know the uniqueness of the smooth solution to (3.4).

For $C \geq \frac{r^2}{n-1}$, one has

$$\mathcal{L}\left(\frac{Ct}{r}\right) = \frac{n-1}{t} \frac{C}{r} \geq \frac{(n-1)C}{r^2} \geq 1 = \mathcal{L}\phi_0.$$

Let $K_1 > 0$ be a constant with $w_{K_1}(r) = C$. Combining (3.2) and Lemma 2.2 gives $\frac{Ct}{r} \leq \phi_0(t) \leq w_{K_1}(t)$ on $[0, r]$.

For $C < 0$ and $r < 1$, one selects a negative constant K_2 satisfying $\tilde{w}_{K_2}(r) = C$. If $K_2^2 \geq \frac{r^{2n+2}}{1-r^2}$, then (3.3) yields

$$(3.10) \quad \mathcal{L}\tilde{w}_{K_2} \geq -\frac{K_2}{r\sqrt{r^{2n} + K_2^2}} \geq 1 = \mathcal{L}\phi_0.$$

By Lemma 2.2, we have $\tilde{w}_{K_2}(t) \leq \phi_0(t) \leq \frac{Ct}{r}$ on $[0, r]$. □

Let Ω be a bounded domain in \mathbb{R}^n and φ be a weakly spacelike function on $\overline{\Omega}$. Let $F_\Omega(\cdot)$ be defined previously. Clearly, $\sup_{w \in \mathcal{C}(\varphi, \Omega)} F_\Omega(w)$ is bounded, which implies there exists a sequence $\{u_k\} \subset \mathcal{C}(\varphi, \Omega)$ such that $\lim_{k \rightarrow \infty} F_\Omega(u_k) = \sup_{w \in \mathcal{C}(\varphi, \Omega)} F_\Omega(w)$. The equicontinuity of $\mathcal{C}(\varphi, \Omega)$ then gives a uniformly convergent subsequence u_{k_i} of maximizing sequence u_k with $u_{k_i} \rightarrow u_0 \in \mathcal{C}(\varphi, \Omega)$. By the Appendix, $F_\Omega(u_0) = \sup_{w \in \mathcal{C}(\varphi, \Omega)} F_\Omega(w)$ is maximal.

The next result for translating solitons is similar to hypersurfaces with bounded mean curvature in Minkowski space, see Theorem 3.2 in [2] for example. However, the mean curvature of any translating soliton is unbounded (see Proposition 4.1).

Lemma 3.2. *If u is a weakly spacelike solution to the variational functional F . Let $x_0, x_1 \in \overline{\Omega}$ with the open line segment $\overline{x_0 x_1} \subset \Omega$ such that*

$$(3.11) \quad u(x_t) = u(x_0) + t|x_0 - x_1| \quad \text{for } t \in [0, 1],$$

where $x_t = x_0 + t(x_1 - x_0)$. Then (3.11) holds for all $t \in \mathbb{R}$ such that $x_t \in \Omega$ and $\overline{x_0 x_t} \subset \Omega$.

Proof. We prove it by following the steps of the proof for Theorem 3.2 in [2]. Suppose that (3.11) holds for $\forall t \in [-\frac{1}{4}, 1]$, $u(x_{-\frac{1}{2}}) > u(x_0) - \frac{1}{2}|x_0 - x_1|$, $|x_0 - x_1| < 1$ and the Euclidean ball $B_{2|x_0 - x_1|}(x_1) \subset \subset \Omega$. Let $B = B_{\frac{1}{2}|x_0 - x_1|}(x_0)$, and let C_1 and C_2 denote the backward light cones with apexes at $(x_1, u(x_1))$ and $(x_0, u(x_0))$ respectively. Then we have

$$u(x) \geq C_1(x) > C_2(x) \quad \text{for any } x \in B \setminus \{x_t \mid t \in [-\frac{1}{2}, 0]\}.$$

Combining $u(x_{-\frac{1}{2}}) > u(x_0) - \frac{1}{2}|x_0 - x_1|$ gives

$$u(x) > C_2(x) \quad \text{for any } x \in \partial B.$$

Let \tilde{w}_K be defined as (3.1) with sufficiently small negative number K , then

$$u(x) > \tilde{w}_K(|x - x_0|) + u(x_0) > C_2(x) \quad \text{for any } x \in \partial B.$$

But \tilde{w}_K is strictly spacelike away from x_0 , so $\tilde{w}_K(|x_t - x_0|) + u(x_0) > u(x_t)$ for $t \in [-\frac{1}{4}, 0]$, and this contradicts Lemma 2.3 applied with $\Omega = B \setminus \{x_0\}$.

A completely analogous argument holds if (3.11) fails for $t > 1$. □

Let u be a smooth function satisfying (2.4) and g_{ij}, g^{kl} be defined as in § 2, then

$$g^{ij}u_{ijk} - g^{ip}(\partial_k g_{pq})g^{qj}u_{ij} = 0.$$

Here we have adopted Einstein convention of summation over repeated indices. Denote $v = \sqrt{1 - |Du|^2}$ as before. Hence

$$\begin{aligned}
 g^{ij}(|Du|^2)_{ij} &= 2g^{ij}(u_{ki}u_{kj} + u_{ijk}u_k) \\
 &= 2g^{ij}u_{ki}u_{kj} - 2g^{ip}(u_pu_{kq} + u_qu_{kp})u_kg^{qj}u_{ij} \\
 &= 2g^{ij}u_{ki}u_{kj} - 4g^{ip}u_pu_{kq}u_kg^{qj}u_{ij} \\
 &= 2g^{ij}u_{ki}u_{kj} - \frac{4}{v^2}u_{kq}u_kg^{qj}u_iu_{ij} \\
 &= 2g^{ij}u_{ki}u_{kj} - \frac{1}{v^2}g^{qj}(|Du|^2)_q(|Du|^2)_j,
 \end{aligned}
 \tag{3.12}$$

i.e.,

$$g^{ij}(|Du|^2)_{ij} + \frac{1}{v^2}g^{ij}(|Du|^2)_i(|Du|^2)_j = 2g^{ij}u_{ki}u_{kj} \geq 0.
 \tag{3.13}$$

4. CONVEXITY AND DIRICHLET PROBLEM

In this section, we always suppose that $M = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ is a smooth strictly spacelike hypersurface in \mathbb{R}_1^{n+1} with u satisfying (1.3). Let ∇ , div_M and Δ be Levi-Civita connection, divergence and Laplacian operator on M with the induced metric from \mathbb{R}_1^{n+1} , respectively. Inspired by the 'drift Laplacian' on self-shrinkers, which was introduced by Colding-Minicozzi [5], we define a second order differential operator L by

$$Lf = e^u \text{div}_M(e^{-u} \nabla f) \quad \text{for } f \in C^2(\mathbb{R}^n).$$

Let E_1, \dots, E_{n+1} be the unit natural basis of \mathbb{R}_1^{n+1} , and $\partial_j f = \frac{\partial f}{\partial x_j}$. Denote ν be the unit normal vector field of M : $\frac{1}{\sqrt{1-|Du|^2}}(\sum_{i=1}^n u_i E_i + E_{n+1})$. Since

$$\begin{aligned}
 \nabla f &= \sum_{i=1}^n f_i E_i + \left\langle \sum_{i=1}^n f_i E_i, \nu \right\rangle \nu = \sum_{i=1}^n f_i E_i + \frac{1}{\sqrt{1-|Du|^2}} \left(\sum_{i=1}^n u_i f_i \right) \nu \\
 &= \sum_{i=1}^n \left(f_i + \frac{1}{1-|Du|^2} \left(\sum_{j=1}^n u_j f_j \right) u_i \right) E_i + \frac{1}{1-|Du|^2} \left(\sum_{i=1}^n u_i f_i \right) E_{n+1},
 \end{aligned}
 \tag{4.1}$$

then one has

$$\nabla u = \frac{1}{1-|Du|^2} \sum_{i=1}^n u_i E_i + \frac{|Du|^2}{1-|Du|^2} E_{n+1},
 \tag{4.2}$$

and

$$\langle \nabla u, \nabla f \rangle = \frac{1}{1-|Du|^2} \sum_{i=1}^n u_i f_i.
 \tag{4.3}$$

(4.1) and (4.3) imply

$$(4.4) \quad Lf = \Delta f - \langle \nabla u, \nabla f \rangle = \Delta f - \frac{1}{1 - |Du|^2} \sum_{i=1}^n u_i f_i = \Delta f + \langle E_{n+1}, \nabla f \rangle.$$

Proposition 4.1. *The manifold $M = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ has unbounded mean curvature.*

Proof. Suppose that the mean curvature H is bounded, then there exists a positive constant $\delta > 0$ such that $v = \sqrt{1 - |Du|^2} \geq \delta$. Recall that $g_{ij} = \delta_{ij} - u_i u_j$ and $g^{ij} = \delta_{ij} + \frac{u_i u_j}{1 - |Du|^2}$. Let $g = \det g_{ij}$, then by (1.3) one has

$$(4.5) \quad \Delta u = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_i (g^{ij} \sqrt{g} \partial_j u) = \frac{1}{\sqrt{g}} \sum_i \partial_i \left(\frac{u_i}{v} \right) = \frac{1}{v^2}.$$

Let B_r be a ball in \mathbb{R}^n with radius r and centered at the origin. Let η be a nonnegative Lipschitz function with $\eta|_{B_r} \equiv 1$, $|D\eta| \leq \frac{1}{r}$ and $\eta|_{\mathbb{R}^n \setminus B_{2r}} \equiv 0$. For any $p = (x, u(x)) \in M$, we set $\eta(x, u(x)) = \eta(x)$. Denote ω_n be the volume of n -unit ball. Noting $g^{ij} u_j = \frac{u_i}{1 - |Du|^2}$, by (1.3) we have

$$(4.6) \quad \begin{aligned} \omega_n r^n &\leq \int_{B_r} \frac{1}{v} dx \leq \int_{\mathbb{R}^n} \frac{\eta}{v^2} v dx = \int_{\mathbb{R}^n} \eta \partial_i (g^{ij} \sqrt{g} \partial_j u) dx = - \int_{\mathbb{R}^n} \partial_i \eta g^{ij} \partial_j u \sqrt{g} dx \\ &= - \int_{\mathbb{R}^n} \frac{Du \cdot D\eta}{v^2} v dx \leq \int_{\mathbb{R}^n} \frac{|D\eta|}{v} dx \leq \frac{1}{r\delta} \int_{B_{2r}} dx = \frac{1}{\delta} \omega_n 2^n r^{n-1}. \end{aligned}$$

Selecting sufficiently large r , we get the desired contradiction. \square

Proposition 4.2. *If $u \geq 0$ and $\int_M u^2 e^{-u} d\mu < \infty$, then M is noncomplete.*

Proof. Suppose that M is complete, then we could define a nonnegative Lipschitz function η in M satisfying $\eta|_{D_r} \equiv 1$, $|\nabla \eta| \leq \frac{1}{r}$ and $\eta|_{M \setminus D_{2r}} \equiv 0$ for any $r > 0$. Here, D_r is a geodesic ball with radius r in M . By (4.5), we have

$$Lu = \Delta u - |\nabla u|^2 = \frac{1}{v^2} - \frac{|Du|^2}{v^2} = 1,$$

then

$$(4.7) \quad \begin{aligned} \int_{D_r} u e^{-u} &\leq \int_M u \eta^2 e^{-u} = \int_M u \eta^2 e^{-u} Lu = - \int_M \nabla u \cdot \nabla (u \eta^2) e^{-u} \\ &= - \int_M \eta^2 |\nabla u|^2 e^{-u} - 2 \int_M u \eta \nabla u \cdot \nabla \eta e^{-u} \\ &\leq \int_M u^2 |\nabla \eta|^2 e^{-u} \leq \frac{1}{r^2} \int_M u^2 e^{-u}. \end{aligned}$$

Here the volume form $d\mu$ is omitted in the above integrations for notational simplicity.

Passing to the limit as $r \rightarrow +\infty$ we get $u \equiv 0$, but this is not a solution to (1.3). \square

We choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ of M and let $\bar{\nabla}$ be the Levi-Civita connection of \mathbb{R}_1^{n+1} as before. Let B be the second fundamental form and $B_{e_i e_j} = h(e_i, e_j)\nu = h_{ij}\nu$. Then the coefficients of the second fundamental form h_{ij} is a symmetric 2-tensor on M and

$$(4.8) \quad h_{ij} = -\langle \bar{\nabla}_{e_i} e_j, \nu \rangle.$$

By (2.3), mean curvature $H = \sum_i h_{ii} = \frac{1}{\sqrt{1-|Du|^2}}$. Denote the square of the second fundamental form $|B|^2 = \sum_{i,j} h_{ij}^2$.

Lemma 4.3. *Let h_{ij} and L be as defined previously. In the meaning of covariant, we have*

$$(4.9) \quad Lh_{ij} = \Delta h_{ij} + \langle E_{n+1}, \nabla h_{ij} \rangle = |B|^2 h_{ij}.$$

Proof. By Ricci identity, one has

$$(4.10) \quad \Delta h_{ij} = h_{ijkk} = h_{ikjk} = h_{ikkj} + h_{il}R_{klkj} + h_{kl}R_{ilkj}.$$

Combining Gauss formula $R_{ijkl} = -h_{ik}h_{jl} + h_{il}h_{jk}$ (see [6] or [18] for example) and (2.3), we have

$$(4.11) \quad \begin{aligned} \Delta h_{ij} &= H_{ij} + h_{il}(-h_{kk}h_{lj} + h_{kj}h_{kl}) + h_{kl}(-h_{ik}h_{lj} + h_{ij}h_{kl}) \\ &= -(\langle E_{n+1}, \nu \rangle)_{ij} - Hh_{ik}h_{jk} + |B|^2 h_{ij}. \end{aligned}$$

Since

$$(4.12) \quad \begin{aligned} \nabla_{e_i}(h(e_j, e_k)) &= (\nabla_{e_i} h)(e_j, e_k) + h(\nabla_{e_i} e_j, e_k) + h(e_j, \nabla_{e_i} e_k) \\ &= h_{jki} + \langle \nabla_{e_i} e_j, e_l \rangle h_{kl} + \langle \nabla_{e_i} e_k, e_l \rangle h_{jl}, \end{aligned}$$

then

$$(4.13) \quad \begin{aligned} -(\langle E_{n+1}, \nu \rangle)_{ij} &= -\nabla_{e_i} \nabla_{e_j} \langle E_{n+1}, \nu \rangle + \nabla_{\nabla_{e_i} e_j} \langle E_{n+1}, \nu \rangle \\ &= -\nabla_{e_i} (\langle E_{n+1}, e_k \rangle h_{jk}) + \langle \nabla_{e_i} e_j, e_k \rangle \langle E_{n+1}, \bar{\nabla}_{e_k} \nu \rangle \\ &= -\langle E_{n+1}, \bar{\nabla}_{e_i} e_k \rangle h_{jk} - \langle E_{n+1}, e_k \rangle (h_{jki} + \langle \nabla_{e_i} e_j, e_l \rangle h_{kl} \\ &\quad + \langle \nabla_{e_i} e_k, e_l \rangle h_{jl}) + \langle \nabla_{e_i} e_j, e_k \rangle \langle E_{n+1}, e_l \rangle h_{kl} \\ &= -\langle E_{n+1}, \nu \rangle h_{ik} h_{jk} - \langle E_{n+1}, \nabla_{e_i} e_k \rangle h_{jk} - \langle E_{n+1}, e_k \rangle h_{ijk} \\ &\quad - \langle E_{n+1}, e_k \rangle \langle \nabla_{e_i} e_k, e_l \rangle h_{jl} \\ &= -\langle E_{n+1}, \nu \rangle h_{ik} h_{jk} - \langle E_{n+1}, e_l \rangle \langle e_l, \nabla_{e_i} e_k \rangle h_{jk} - \langle E_{n+1}, \nabla h_{ij} \rangle \\ &\quad + \langle E_{n+1}, e_k \rangle \langle e_k, \nabla_{e_i} e_l \rangle h_{jl} \\ &= Hh_{ik} h_{jk} - \langle E_{n+1}, \nabla h_{ij} \rangle. \end{aligned}$$

Combining (4.4), (4.11) and (4.13), we complete the Lemma. \square

Let u be a smooth solution to (1.3) in \mathbb{R}^n . For any constant $h > \inf_{x \in \mathbb{R}^n} u(x)$, we denote

$$(4.14) \quad \begin{aligned} \Gamma_{h,u} &= \{x \in \mathbb{R}^n \mid u(x) = h\}, \\ \Omega_{h,u} &= \{x \in \mathbb{R}^n \mid u(x) < h\}. \end{aligned}$$

and call them the level set and the sublevel set of u , respectively.

Now we restrict $\Gamma_{h,u}$ as a hypersurface in \mathbb{R}^n . Let γ be the unit outward normal vector of $\Gamma_{h,u}$ and Δ_Γ be the Laplacian operator of $\Gamma_{h,u}$. Let ∇^Γ and $\nabla^{\mathbb{R}^n}$ be the Levi-Civita connections of $\Gamma_{h,u}$ and \mathbb{R}^n , respectively. If $\{\beta_i\}_{i=1}^{n-1}$ is an orthonormal frame of $\Gamma_{h,u}$, then the mean curvature of $\Gamma_{h,u}$

$$H_\Gamma \triangleq - \sum_i \langle \nabla_{\beta_i}^{\mathbb{R}^n} \beta_i, \gamma \rangle.$$

At any point of $\Gamma_{h,u}$,

$$(4.15) \quad \begin{aligned} \Delta_{\mathbb{R}^n} u &= \sum_i (\beta_i \beta_i u - (\nabla_{\beta_i}^{\mathbb{R}^n} \beta_i) u) + u_{\gamma\gamma} = \sum_i (\beta_i \beta_i u - (\nabla_{\beta_i}^\Gamma \beta_i) u) + H_\Gamma u_\gamma + u_{\gamma\gamma} \\ &= \Delta_\Gamma u + H_\Gamma u_\gamma + u_{\gamma\gamma} = H_\Gamma u_\gamma + u_{\gamma\gamma}, \end{aligned}$$

and

$$(4.16) \quad u_i u_j u_{ij} = \frac{1}{2} u_i \partial_i (|Du|^2) = \frac{1}{2} Du \cdot Du_\gamma^2 = \frac{1}{2} \partial_\gamma u \cdot \partial_\gamma u_\gamma^2 = u_\gamma^2 u_{\gamma\gamma}.$$

Then combining (2.4)(4.15)(4.16), we have (compared to the Euclidean case[20])

$$(4.17) \quad H_\Gamma u_\gamma + \frac{u_{\gamma\gamma}}{1 - u_\gamma^2} = 1.$$

Lemma 4.4. *If $\Gamma_{h,u}$ is a nonempty convex compact set in \mathbb{R}^n with mean curvature $H_\Gamma \leq 1$, then u is convex in $\Omega_{h,u}$.*

Proof. Since $\Gamma_{h,u}$ is convex, then $0 \leq H_\Gamma \leq 1$. Combining $|u_\gamma| \leq 1$ and (4.17), we have

$$(4.18) \quad u_{\gamma\gamma} = (1 - u_\gamma^2)(1 - H_\Gamma u_\gamma) \geq 0,$$

which implies $D^2 u|_{\Gamma_{h,u}} \geq 0$. Let $\frac{\partial}{\partial x_i} = E_i + u_i E_{n+1}$ be a tangential vector field in M , then

$$(4.19) \quad h \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = - \left\langle \bar{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \nu \right\rangle = -u_{ij} \langle E_{n+1}, \nu \rangle = \frac{u_{ij}}{\sqrt{1 - |Du|^2}}.$$

Let $\lambda(x)$ be the minimal principal curvature of the second fundamental form at the point $x \in M$ (see [6] or [19] for Ricci curvature). Then (4.19) implies $\lambda|_{\Gamma_{h,u}} \geq 0$. If $\lambda(x)$ is not nonnegative in $\Omega_{h,u}$, then there is a point $p_0 \in \Omega_{h,u}$ such that $\lambda(p_0) = \inf_{x \in \Omega_{h,u}} \lambda(x) <$

0. Choose a local orthonormal frame $\{e_i\}$ near p_0 in M and denote $h_{ij} = h(e_i, e_j)$ as mentioned before. Let $\theta = \sum_i \theta_i e_i|_{p_0}$ be a unit eigenvector of the second fundamental form with eigenvalue $\lambda(p_0)$ at the point p_0 , namely, $h(\theta, \theta) = \lambda(p_0)$. Then we define a smooth function by

$$f(x) = \sum_{i,j} h_{ij} \Big|_x \theta_i \theta_j.$$

f attains the minimal value $\lambda(p_0)$ in a neighborhood of p_0 . At the point p_0 , by (4.9) we have

$$(4.20) \quad 0 \leq Lf = L(h_{ij}\theta_i\theta_j) = L(h_{ij})\theta_i\theta_j = |B|^2 h_{ij}\theta_i\theta_j = |B|^2 f \leq \frac{H^2}{n} f < 0.$$

This is a contradiction. Therefore, (h_{ij}) is nonnegative in $\overline{\Omega}_{h,u}$, which yields the Lemma. \square

Lemma 4.5. *Let Ω be a bounded convex domain with smooth boundary in \mathbb{R}^n and $\Omega_\sigma = \{\sigma x \mid x \in \Omega\}$ for $\sigma \in (0, 1]$. If u_σ is a smooth strictly spacelike solution to (2.4) in Ω_σ with $u_\sigma|_{\partial\Omega_\sigma} = 0$, then there is a constant $\theta \in (0, 1)$ depending only on the diameter of $\overline{\Omega}$, such that $\max_{\overline{\Omega}_\sigma} |Du_\sigma| \leq 1 - \theta$.*

Proof. Let $d = \text{diam}(\overline{\Omega})$ be the diameter of $\overline{\Omega}$. For any fixed $y \in \partial\Omega_\sigma$, we assume $y = (y_1, 0, \dots, 0)$ and $\Omega_\sigma \subset \{x = (x_1, \dots, x_n) \mid -y_1 < x_1 < y_1\}$ with $0 < y_1 \leq \frac{d}{2}$. Let $\varphi(x) = \log \cosh(x_1) - \log \cosh(y_1)$, then $\mathcal{L}\varphi = 1$, $\varphi(y) = 0$ and $\varphi|_{\partial\Omega_\sigma} \leq 0$. Using Lemma 2.2 for $\pm\varphi$, one gets

$$\varphi(x) \leq u_\sigma(x) \leq -\varphi(x) \quad \text{for } x \in \Omega_\sigma.$$

Combining $u_\sigma|_{\partial\Omega_\sigma} = \varphi(y) = 0$ gives

$$|Du_\sigma(y)| \leq |D\varphi(y)| \leq \tanh d.$$

By the maximum principle for (3.13), we complete the proof. \square

Now let us consider a Dirichlet problem, which may be inconvenient to be found out directly in general theories of PDE.

Theorem 4.6. *Let Ω be a bounded convex domain with smooth boundary in \mathbb{R}^n . Then the Dirichlet problem $\mathcal{L}u = 1$ in Ω with $u|_{\partial\Omega} = 0$ has a smooth strictly spacelike solution u on $\overline{\Omega}$.*

Proof. For $f \in C^{2,\alpha}(\overline{\Omega})$, $|Df| < 1$ and $\sigma \in (0, 1]$, we define Q_σ by

$$(4.21) \quad Q_\sigma f = \sum_{i,j} \left(\delta_{ij} + \frac{f_i f_j}{1 - |Df|^2} \right) f_{ij} - \sigma.$$

If $u_\sigma \in C^{2,\alpha}(\overline{\Omega})$ is a strictly spacelike solution to $Q_\sigma u_\sigma = 0$ in Ω and $u_\sigma|_{\partial\Omega} = 0$, then $u_\sigma \in C^\infty(\overline{\Omega})$ by standard regularity theory of elliptic equations. Let $\Omega_\sigma = \{\sigma x \mid x \in \Omega\}$ and

$$(4.22) \quad u(x) = \sigma u_\sigma \left(\frac{x}{\sigma} \right),$$

then u is a smooth strictly spacelike solution to (2.4) in Ω_σ .

By Lemma 4.5, there is a constant $\theta \in (0, 1)$ depending only on the diameter of $\overline{\Omega}$, such that $\max_{\overline{\Omega}_\sigma} |Du| \leq 1 - \theta$. If the mean curvature of $\partial\Omega$ satisfies $0 \leq H_{\partial\Omega} \leq C_1$ for some $C_1 \geq 1$, then

$$0 \leq H_{\partial\Omega_\sigma} \leq \frac{C_1}{\sigma}.$$

Let γ be the unit outward normal vector of $\partial\Omega_\sigma$, then by (4.18), we have $1 - \frac{C_1}{\sigma} \leq u_{\gamma\gamma} \leq 1$. Hence,

$$D^2u + \left(\frac{C_1}{\sigma} - 1 \right) I$$

is positive definite on $\partial\Omega_\sigma$, where I is a unit $n \times n$ matrix. Moreover, $1 - \frac{C_1}{\sigma} \leq u_{ii} \leq C_2$ for some C_2 depending only on n, Ω and

$$(u_{ij})^2 \leq \left(u_{ii} + \frac{C_1}{\sigma} - 1 \right) \left(u_{jj} + \frac{C_1}{\sigma} - 1 \right) \leq \left(C_2 + \frac{C_1}{\sigma} - 1 \right)^2 \quad \text{for } i \neq j.$$

Let h_{ij} be defined as (4.8), where we replace M by $\{(x, u(x)) \mid x \in \Omega_\sigma\}$. By the proof of Lemma 4.4, the matrix (h_{ij}) must attain its negative minimal eigenvalue and positive maximal eigenvalue on the boundary $\partial\Omega_\sigma$. Combining (4.19) and $\max_{\overline{\Omega}_\sigma} |Du| \leq 1 - \theta$, we get that there is a constant C_3 depending only on n, Ω such that

$$\max_{x \in \overline{\Omega}_\sigma} u_{ij}(x) \leq \frac{C_3}{\sigma} \quad \text{for } i, j = 1, \dots, n.$$

(4.22) implies $\max_{\overline{\Omega}} \partial_{ij} u_\sigma \leq C_3$. For some $\beta \in (0, 1)$ there is a constant C_β depending only on n, β, Ω such that

$$(4.23) \quad |u_\sigma|_{C^{1,\beta}(\overline{\Omega})} \leq C_\beta \quad \text{and} \quad \max_{\overline{\Omega}} |Du_\sigma| \leq 1 - \theta.$$

For any $f \in C^{1,\beta}(\overline{\Omega})$ with $\max_{\overline{\Omega}} |Df| < 1$, the operator T is defined by letting $u = Tf$ be the unique solution in $C^{2,\alpha,\beta}(\overline{\Omega})$ of the linear Dirichlet problem (see [12] for example):

$$\sum_{i,j} \left(\delta_{ij} + \frac{f_i f_j}{1 - |Df|^2} \right) u_{ij} - 1 = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

If $|\cdot|_{C^{1,\beta}(\overline{\Omega})}$ is a norm in Hölder space $C^{1,\beta}(\overline{\Omega})$, then we may define a new norm $\|\cdot\|$ in $C^{1,\beta}(\overline{\Omega})$ by

$$\|f\| = \max_{\overline{\Omega}} |Df| + \frac{\theta}{2C_\beta} |f|_{C^{1,\beta}(\overline{\Omega})}.$$

Let $\mathfrak{S} = \{f \in C^{1,\beta}(\overline{\Omega}) \mid \|f\| \leq 1 - \frac{\theta}{4}\}$ be a closed ball in $C^{1,\beta}(\overline{\Omega})$. We define a mapping T^* by

$$T^*f \triangleq \begin{cases} Tf & \text{if } \|Tf\| \leq 1 - \frac{\theta}{4} \\ (1 - \frac{\theta}{4}) \frac{Tf}{\|Tf\|} & \text{if } \|Tf\| \geq 1 - \frac{\theta}{4}, \end{cases}$$

then T^* is a mapping from \mathfrak{S} to \mathfrak{S} . By Arzela's theorem, T^* is a compact operator in \mathfrak{S} . By Corollary 11.2 in [12], T^* has a fixed point w in \mathfrak{S} . If $\|Tw\| \geq 1 - \frac{\theta}{4}$, then $w = T^*w = \sigma Tw$ if $\sigma = (1 - \frac{\theta}{4}) \frac{1}{\|Tw\|}$, and $\|w\| = \|T^*w\| = 1 - \frac{\theta}{4}$. The definition of T implies

$$\sum_{i,j} \left(\delta_{ij} + \frac{w_i w_j}{1 - |Dw|^2} \right) w_{ij} - \sigma = 0 \quad \text{in } \Omega \quad \text{and} \quad w|_{\partial\Omega} = 0.$$

By (4.23), we have $\|w\| \leq 1 - \theta + \frac{\theta}{2C_\beta} C_\beta = 1 - \frac{\theta}{2}$, which is a contradiction. Hence, $\|Tw\| < 1 - \frac{\theta}{4}$ and $w = Tw$. By the standard regularity theory of elliptic equations, w is our desired function. \square

5. ENTIRE SPACELIKE TRANSLATING SOLITONS

By [8], the elliptic equation (1.3) has a radially symmetric solution $\psi(r)$, where $\psi(r) \in C^2([0, +\infty))$ satisfies the following ODE

$$(5.1) \quad \frac{\psi''}{1 - (\psi')^2} + \frac{n-1}{r} \psi' = 1 \quad \text{for } r \in (0, +\infty)$$

with $\psi'(0) = 0$ in an appropriate coordinate system. By [16], up to an additive constant (5.1) has a unique smooth convex solution ψ with $\frac{r}{\sqrt{n^2 + r^2}} \leq \psi' < 1$ for $r \geq 0$. Thus, for $r \geq 0$ one has

$$(5.2) \quad r - n \leq \int_0^r \frac{s}{\sqrt{n^2 + s^2}} ds \leq \int_0^r \psi'(s) ds = \psi(r) - \psi(0) \leq r.$$

Lemma 5.1. *Let Ω_k be a family of convex domains with smooth boundaries ($\partial\Omega_k$ may be empty), $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\bigcup_{k \geq 1} \Omega_k = \mathbb{R}^n$. Let u_k be a smooth convex strictly spacelike function to (2.4) in Ω_k . Denote $x_t = x + t(y - x)$ for $x, y \in \mathbb{R}^n$. Suppose that there is a function W in \mathbb{R}^n satisfying*

$$\limsup_{t \rightarrow +\infty} \frac{|W(x_t) - W(x_{-t})|}{|x_t - x_{-t}|} < 1$$

for any $x \neq y \in \mathbb{R}^n$. If $\limsup_{k \rightarrow \infty} |u_k(x) - W(x)| \leq C$ for some absolute constant C and any $x \in \mathbb{R}^n$, then there is a subsequence of $\{u_k\}$ converging to an entire smooth convex strictly spacelike function u to (2.4) in \mathbb{R}^n .

Proof. By convexity of u_k and (2.4), we have

$$1 = \sum_{i=1}^n \left(\delta_{ij} + \frac{\partial_i u_k \partial_j u_k}{1 - |Du_k|^2} \right) \partial_{ij} u_k \geq \sum_i \partial_{ii} u_k.$$

For any $i, j = 1, \dots, n$ one gets

$$(\partial_{ij} u_k)^2 \leq \partial_{ii} u_k \partial_{jj} u_k \leq \left(\frac{\partial_{ii} u_k + \partial_{jj} u_k}{2} \right)^2 \leq 1.$$

There is a subsequence $\{u_{k_j}\}$ of $\{u_j\}$ converging to u in $C^1(K)$ uniformly for any compact set $K \subset \mathbb{R}^n$. Then $u \in C^{1,1}(\mathbb{R}^n)$ is a convex function with $|Du| \leq 1$. Clearly,

$$W(x) - C \leq u(x) \leq W(x) + C \quad \text{for each } x \in \mathbb{R}^n.$$

Let Ω be an arbitrary bounded domain in \mathbb{R}^n . By the definition of $F_\Omega(\cdot)$ in § 2 and the Appendix, there is a weakly spacelike function $w_0 \in \mathcal{C}(u, \Omega)$ such that

$$F_\Omega(w_0) = \sup_{w \in \mathcal{C}(u, \Omega)} F_\Omega(w).$$

For any $\epsilon > 0$, denote $\Omega_{j,\epsilon}^+ = \{x \in \Omega \mid u_{k_j}(x) > w_0(x) - \epsilon\}$ and $\Omega_\epsilon^+ = \{x \in \Omega \mid u(x) > w_0(x) - \epsilon\}$. By Theorem 2.1 one has

$$F_{\Omega_{j,\epsilon}^+}(w_0 - \epsilon) \leq F_{\Omega_{j,\epsilon}^+}(u_{k_j}) \leq F_{\Omega_\epsilon^+}(u_{k_j}) + F_{\Omega_{j,\epsilon}^+ \setminus \Omega_\epsilon^+}(u_{k_j}).$$

If the set $\Gamma_\epsilon = \{x \in \Omega \mid u(x) = w_0(x) - \epsilon\}$ has positive n -dimensional Lebesgue measure, then for any open set $U_\epsilon \subset \Gamma_\epsilon$, $F_{U_\epsilon}(w_0 - \epsilon) = F_{U_\epsilon}(u)$. Since $u_{k_j} \rightarrow u$ in C^1 -norm, then

$$\Omega_\epsilon^+ \subset \liminf_{j \rightarrow \infty} \Omega_{j,\epsilon}^+ \subset \limsup_{j \rightarrow \infty} \Omega_{j,\epsilon}^+ \subset \Omega_\epsilon^+ \cup \Gamma_\epsilon.$$

Letting $j \rightarrow \infty$ gives

$$(5.3) \quad e^\epsilon F_{\Omega_\epsilon^+}(w_0) = F_{\Omega_\epsilon^+}(w_0 - \epsilon) \leq F_{\Omega_\epsilon^+}(u).$$

If the set $\Gamma = \{x \in \Omega \mid u(x) = w_0(x)\}$ has positive n -dimensional Lebesgue measure, then for any open set $U \subset \Gamma$, $F_U(w_0) = F_U(u)$. Denote $\Omega^+ = \{x \in \Omega \mid u(x) > w_0(x)\}$ and $\Omega^- = \{x \in \Omega \mid u(x) < w_0(x)\}$, then

$$\Omega^+ \subset \liminf_{\epsilon \rightarrow 0} \Omega_\epsilon^+ \subset \limsup_{\epsilon \rightarrow 0} \Omega_\epsilon^+ \subset \Omega^+ \cup \Gamma.$$

Let $\epsilon \rightarrow 0$ in (5.3), we obtain

$$F_{\Omega^+}(w_0) \leq F_{\Omega^+}(u).$$

By the same way, one gets $F_{\Omega^-}(w_0) \leq F_{\Omega^-}(u)$. $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$ implies

$$F_\Omega(w_0) = F_{\Omega^+}(w_0) + F_{\Omega^-}(w_0) + F_\Gamma(w_0) \leq F_\Omega(u).$$

Thus the definition of w_0 tells us that u is a weak solution to the variational functional F in any bounded domain Ω .

If there is $x \neq y \in \mathbb{R}^n$ such that $|u(x) - u(y)| = |x - y|$. Then by Lemma 3.2, we obtain

$$(5.4) \quad |u(x_t) - u(x_{-t})| = |x_t - x_{-t}| \quad \text{for } \forall t \in \mathbb{R},$$

where $x_t = x + t(y - x)$. The definition of W and $W(x) - C \leq u(x) \leq W(x) + C$ imply that (5.4) is impossible. Hence $|u(x) - u(y)| < |x - y|$ for any $x \neq y$. Noting the convexity of $u \in C^{1,1}(\mathbb{R}^n)$, we get $|Du| < 1$. The regularity theory of elliptic equations and $u \in C^{1,1}(\mathbb{R}^n)$ force that u is a smooth solution to (1.3). \square

Let Q be defined as in § 1 and Q_0 be the set that contains all linear functions through the origin whose gradient has norm one. Denote \mathbb{S}^{n-1} be the unit sphere centered at the origin, then for any $V \in Q \setminus Q_0$ there is a closed set $\Lambda \subset \mathbb{S}^{n-1}$ containing two points at least, such that $V(x) = \sup_{\lambda \in \Lambda} \langle \lambda, x \rangle$ (see [4] for example).

Theorem 5.2. *For $n \geq 2$ and any $V \in Q \setminus Q_0$ there is an entire smooth convex strictly spacelike solution u to (1.3) such that u blows down to V .*

Proof. (of Theorem 1.1). For any fixed positive constant K and $V \in Q \setminus Q_0$, let $\tilde{V}(x) \triangleq \max\{V(x), |x| - K\}$, then

$$\lim_{|x| \rightarrow \infty} \frac{\tilde{V}(x)}{|x|} = 1.$$

Clearly, $\tilde{V}(x)$ is a convex weakly spacelike function in \mathbb{R}^n with $\tilde{V}(0) = 0$. Let \tilde{V}_ϵ be a mollifier of \tilde{V} defined by

$$(5.5) \quad \tilde{V}_\epsilon(x) = \int_{\mathbb{R}^n} \rho(y) \tilde{V}(x - \epsilon y) dy = \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \rho\left(\frac{x - y}{\epsilon}\right) \tilde{V}(y) dy,$$

where ϵ is a positive constant to be determined below and ρ is the function defined in § 2. Then

$$(5.6) \quad |\tilde{V}_\epsilon(x) - \tilde{V}(x)| = \left| \int_{\mathbb{R}^n} \rho(y) (\tilde{V}(x - \epsilon y) - \tilde{V}(x)) dy \right| \leq \int_{\mathbb{R}^n} \rho(y) \epsilon |y| dy \leq \epsilon,$$

and $|D\tilde{V}_\epsilon(x)| \leq 1$ by (2.6). Denote $\Omega_{h, \tilde{V}_\epsilon} = \{x \in \mathbb{R}^n \mid \tilde{V}_\epsilon(x) < h\}$ and $\Gamma_{h, \tilde{V}_\epsilon} = \partial\Omega_{h, \tilde{V}_\epsilon}$. The convexity of \tilde{V}_ϵ implies that $\Omega_{h, \tilde{V}_\epsilon}$ is convex. Let $\{\xi_1, \dots, \xi_n\}$ be an orthonormal coordinate transform of $\{x_1, \dots, x_n\}$ such that $\frac{\partial}{\partial \xi_n}$ is the outward normal vector of $\Gamma_{h, \tilde{V}_\epsilon}$. For sufficiently large $h > 0$ and each $x \in \Gamma_{h, \tilde{V}_\epsilon}$,

$$(5.7) \quad \partial_{\xi_n} \tilde{V}_\epsilon = |D\tilde{V}_\epsilon| \geq \frac{\partial}{\partial r} \tilde{V}_\epsilon \geq \frac{\tilde{V}_\epsilon(x) - \tilde{V}_\epsilon(0)}{|x|} \geq \frac{\tilde{V}(x) - 2\epsilon}{|x|} \geq \frac{1}{2},$$

where we have used the convexity of \tilde{V}_ϵ in the second inequality of (5.7).

Since

$$(5.8) \quad \begin{aligned} \partial_{ij} \tilde{V}_\epsilon(x) &= \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \partial_{x_i x_j} \left(\rho \left(\frac{x-y}{\epsilon} \right) \right) \tilde{V}(y) dy = -\frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \partial_{y_j} \partial_{x_i} \left(\rho \left(\frac{x-y}{\epsilon} \right) \right) \tilde{V}(y) dy \\ &= \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \partial_{x_i} \left(\rho \left(\frac{x-y}{\epsilon} \right) \right) \partial_{y_j} \tilde{V}(y) dy, \end{aligned}$$

then one has

$$(5.9) \quad |\partial_{ij} \tilde{V}_\epsilon(x)| \leq \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \left| \partial_{x_i} \left(\rho \left(\frac{x-y}{\epsilon} \right) \right) \right| dy \leq \frac{1}{\epsilon} \int_{B_1} |D\rho(x)| dx = \frac{C_4}{\epsilon}.$$

Here, C_4 is a constant depending only on n .

For any fixed $y \in \Gamma_{h, \tilde{V}_\epsilon}$, selecting $\{\xi_1, \dots, \xi_{n-1}\}$ such that

$$(\partial_{\xi_i \xi_j} \xi_n)_{(n-1) \times (n-1)}|_y = \text{diag}\{\lambda_1, \dots, \lambda_{n-1}\}.$$

Since $\Omega_{h, \tilde{V}_\epsilon}$ is convex, then $\lambda_i \leq 0$ and $\partial_{\xi_i} \xi_n|_y = 0$ for $i = 1, \dots, n-1$. Taking derivations of the equation $\tilde{V}_\epsilon(x) = h$ at y gives

$$(5.10) \quad \partial_{\xi_i} \tilde{V}_\epsilon + \partial_{\xi_n} \tilde{V}_\epsilon \partial_{\xi_i} \xi_n = 0,$$

and

$$(5.11) \quad \partial_{\xi_i \xi_j} \tilde{V}_\epsilon + \partial_{\xi_i \xi_n} \tilde{V}_\epsilon \partial_j \xi_n + \partial_{\xi_j \xi_n} \tilde{V}_\epsilon \partial_{\xi_i} \xi_n + \partial_{\xi_n} \tilde{V}_\epsilon \partial_{\xi_i \xi_j} \xi_n + \partial_{\xi_n \xi_n} \tilde{V}_\epsilon \partial_{\xi_i} \xi_n \partial_{\xi_j} \xi_n = 0.$$

From $\partial_{\xi_i} \xi_n|_y = 0$, one gets

$$(5.12) \quad \partial_{\xi_i \xi_j} \tilde{V}_\epsilon = -\partial_{\xi_n} \tilde{V}_\epsilon \partial_{\xi_i \xi_j} \xi_n = -\partial_{\xi_n} \tilde{V}_\epsilon \lambda_i \delta_{ij}.$$

The convexity of \tilde{V}_ϵ implies

$$\sum_{i=1}^{n-1} \partial_{\xi_i \xi_i} \tilde{V}_\epsilon \leq \sum_{i=1}^n \partial_{\xi_i \xi_i} \tilde{V}_\epsilon = \sum_{i=1}^n \partial_{ii} \tilde{V}_\epsilon \leq \frac{nC_4}{\epsilon}.$$

Then for sufficiently large h the mean curvature of $\Gamma_{h, \tilde{V}_\epsilon}$ satisfies

$$\begin{aligned} 0 \leq H_{\Gamma_{h, \tilde{V}_\epsilon}} &= -\frac{1}{\sqrt{1 + \sum_l (\partial_{\xi_l} \xi_n)^2}} \sum_{i,j=1}^{n-1} \left(\delta_{ij} - \frac{\partial_{\xi_i} \xi_n \partial_{\xi_j} \xi_n}{1 + \sum_l (\partial_{\xi_l} \xi_n)^2} \right) \partial_{\xi_i \xi_j} \xi_n \\ (5.13) \quad &= -\sum_{i=1}^{n-1} \partial_{\xi_i \xi_i} \xi_n = -\sum_i \lambda_i = \sum_{i=1}^{n-1} \frac{\partial_{\xi_i \xi_i} \tilde{V}_\epsilon}{\partial_{\xi_n} \tilde{V}_\epsilon} \leq \frac{nC_4}{\epsilon \partial_{\xi_n} \tilde{V}_\epsilon} \leq \frac{2nC_4}{\epsilon}. \end{aligned}$$

Let $\epsilon = 2nC_4$, and denote $\Omega_k = \Omega_{k, \tilde{V}_\epsilon}$ for all sufficiently large k . Then we get a family of convex domains $\{\Omega_k\}$ with smooth boundaries and $0 \leq H_{\partial\Omega_k} \leq 1$.

By Theorem 4.6, there is a smooth solution u_k to (2.4) in $\bar{\Omega}_k$ with $u_k|_{\partial\Omega_k} = k$ and $|Du_k| < 1$ in $\bar{\Omega}_k$. Since $|H_{\partial\Omega_k}| \leq 1$, then by Lemma 4.4, u_k is a smooth convex strictly spacelike function in Ω_k . By (5.6), $|\tilde{V}(x) - \tilde{V}_\epsilon(x)| \leq \epsilon$, then for any $x \in \partial\Omega_k$, we have

$$(5.14) \quad \tilde{V}(x) - \epsilon \leq u_k(x) = k \leq \tilde{V}(x) + \epsilon.$$

Let ψ be the solution of (5.1) with $\psi'(0) = 0$ and $\psi(0) = 0$. Combining (5.2) and $|D\tilde{V}| \leq 1$ a.e. gives

$$(5.15) \quad \tilde{V}(x) - \tilde{V}(y) \leq |x - y| \leq \psi(|x - y|) - \psi(0) + n = \psi(|x - y|) + n.$$

Combining (5.14) and (5.15) implies $u_k(x) \leq \tilde{V}(x) + \epsilon \leq \tilde{V}(y) + \psi(|x - y|) + n + \epsilon$ for $x \in \partial\Omega_k$. By Lemma 2.3, we have $u_k(x) \leq \tilde{V}(y) + \psi(|x - y|) + n + \epsilon$ for $x \in \Omega_k$ and each fixed $y \in \mathbb{R}^n$. Hence,

$$(5.16) \quad u_k(x) \leq \inf_{y \in \mathbb{R}^n} \{ \tilde{V}(y) + \psi(|x - y|) \} + n + \epsilon \leq \tilde{V}(x) + n + \epsilon \quad \text{for } x \in \Omega_k.$$

There is a closed set $\Lambda \subset \mathbb{S}^{n-1}$ containing two points at least, such that $V(x) = \sup_{\lambda \in \Lambda} \langle \lambda, x \rangle$. Then (5.14) implies $\langle \lambda, x \rangle - \epsilon \leq u_k(x)$ on $\partial\Omega_k$ for every $\lambda \in \Lambda$. Applying Lemma 2.3 we get $\langle \lambda, x \rangle - \epsilon \leq u_k(x)$ in Ω_k . Similarly, for each $\tilde{\lambda} \in \mathbb{S}^{n-1}$, $\langle \tilde{\lambda}, x \rangle - K - \epsilon \leq u_k(x)$ in Ω_k . Hence $\tilde{V}(x) - \epsilon \leq u_k(x)$ in Ω_k by $|x| = \sup_{\tilde{\lambda} \in \mathbb{S}^{n-1}} \langle \tilde{\lambda}, x \rangle$. Combining (5.16), we obtain

$$(5.17) \quad \tilde{V}(x) - \epsilon \leq u_k(x) \leq \tilde{V}(x) + n + \epsilon \quad \text{for } x \in \Omega_k.$$

There is a subsequence of $\{u_k\}$ converging to a convex weakly spacelike function u_K uniformly in any compact set in \mathbb{R}^n . Clearly, $\tilde{V}(x) - \epsilon \leq u_K(x) \leq \tilde{V}(x) + n + \epsilon$ for $x \in \mathbb{R}^n$.

By Lemma 5.1 and the definition of \tilde{V} , we know that u_K is an entire smooth convex strictly spacelike solution to (1.3).

Note that $\tilde{V}(x) = \max\{V(x), |x| - K\}$ depends on K . Let $\{K_i\}$ be a sequence in \mathbb{R}^+ satisfying $\lim_{i \rightarrow \infty} K_i = +\infty$. Let u_{K_i} be an entire smooth convex strictly spacelike function to (1.3) with

$$\max\{V(x), |x| - K_i\} - \epsilon \leq u_{K_i}(x) \leq \max\{V(x), |x| - K_i\} + n + \epsilon \quad \text{in } \mathbb{R}^n.$$

By Lemma 5.1 and the definition of V , there is a subsequence of $\{u_{K_i}\}$ converging to an entire smooth convex strictly spacelike function u satisfying (1.3) in \mathbb{R}^n . Moreover, $V(x) - \epsilon \leq u(x) \leq V(x) + n + \epsilon$ for $x \in \mathbb{R}^n$, which implies

$$\lim_{r \rightarrow \infty} \frac{u(rx)}{r} = V(x).$$

Hence u blows down to V . □

By Theorem 1.2 in [16], if u is an entire smooth convex strictly spacelike solution to (1.3), then u must blow down to some function V in Q . Namely, V is a convex homogeneous of degree one function such that $|DV(y)| = 1$ for any $y \in \mathbb{R}^n$ where the gradient exists.

Now, let us explore the extent of non-uniqueness for spacelike solutions for given projective data at infinity. Compared to Theorem 2 in [19], we obtain a strong result for translating solitons.

Theorem 5.3. *Let $n \geq 2$ and $f \in C^2(\mathbb{S}^{n-1})$. Then there exists a unique entire smooth convex strictly spacelike solution u to (1.3) such that*

$$\lim_{|x| \rightarrow \infty} \left(u(x) - |x| - f\left(\frac{x}{|x|}\right) \right) = 0.$$

Proof. By Theorem 2 in [19], there are C^1 functions p_1, p_2 on \mathbb{S}^{n-1} such that

$$\langle p_1(\eta), \xi - \eta \rangle \leq f(\xi) - f(\eta) \leq \langle p_2(\eta), \xi - \eta \rangle \quad \text{for } \xi, \eta \in \mathbb{S}^{n-1}.$$

Let ψ be the smooth function satisfying (5.1) with $\psi'(0) = 0$ and $\lim_{r \rightarrow \infty} (\psi(r) - r) = 0$. Let $z_i(x, \eta) = f(\eta) - \langle p_i(\eta), \eta \rangle + \psi(|x + p_i(\eta)|)$ for $x \in \mathbb{R}^n$, $\eta \in \mathbb{S}^{n-1}$ and $i = 1, 2$. By $\lim_{r \rightarrow \infty} (\psi(r\xi + p_i(\eta)) - r) = \langle p_i(\eta), \xi \rangle$ uniformly for $\xi, \eta \in \mathbb{S}^{n-1}$, we have

$$\begin{aligned} \lim_{r \rightarrow \infty} (z_1(r\xi, \eta) - r) &= f(\eta) - \langle p_1(\eta), \eta \rangle + \lim_{r \rightarrow \infty} (\psi(r\xi + p_1(\eta)) - r) \\ (5.18) \quad &= f(\eta) - \langle p_1(\eta), \eta \rangle + \langle p_1(\eta), \xi \rangle \leq f(\xi). \end{aligned}$$

Similarly, we have $f(\xi) \leq \lim_{r \rightarrow \infty} (z_2(r\xi, \eta) - r)$, where equality holds when $\xi = \eta$. Let $q_1(x) = \sup_{\eta \in \mathbb{S}^{n-1}} z_1(x, \eta)$ and $q_2(x) = \inf_{\eta \in \mathbb{S}^{n-1}} z_2(x, \eta)$, then $\lim_{r \rightarrow \infty} (q_i(r\xi) - r) = f(\xi)$ uniformly for $\xi \in \mathbb{S}^{n-1}$ and $i = 1, 2$. Hence, there is a continuous function $s(r) > 0$ with $s(r) \rightarrow 0$ as $r \rightarrow \infty$ such that

$$(5.19) \quad q_1(r\xi) - s(r) \leq r + f(\xi) \leq q_2(r\xi) + s(r) \quad \text{for any } \xi \in \mathbb{S}^{n-1}.$$

We extend f to \tilde{f} by $\tilde{f}(x) = f(\frac{x}{|x|})$ for $|x| \geq 1$. A simply computation shows

$$(5.20) \quad \begin{aligned} \partial_{ij}(|x| + \tilde{f}(x)) &= \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} + \sum_{k,l} \frac{f_{kl}}{|x|^2} \left(\delta_{ki} - \frac{x_k x_i}{|x|^2} \right) \left(\delta_{lj} - \frac{x_l x_j}{|x|^2} \right) \\ &\quad - \frac{x_i f_j + x_j f_i}{|x|^3} - \frac{\sum_k x_k f_k}{|x|^3} \left(\delta_{ij} - 3 \frac{x_i x_j}{|x|^2} \right). \end{aligned}$$

For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{S}^{n-1}$ with $\langle \alpha, x \rangle = 0$ and sufficiently large $|x|$, we have $\sum_{i,j} \partial_{ij}(|x| + \tilde{f}(x)) x_i x_j = 0$ and

$$\frac{1}{2|x|} \leq \sum_{i,j} \partial_{ij}(|x| + \tilde{f}(x)) \alpha_i \alpha_j \leq \frac{2}{|x|}.$$

Hence, the matrix $(\partial_{ij}(|x| + \tilde{f}(x)))_{n \times n}$ could be diagonal to $\text{diag}\{0, \mu_1, \dots, \mu_{n-1}\}$ with $\frac{1}{2|x|} \leq \mu_i \leq \frac{2}{|x|}$ for $1 \leq i \leq n-1$ and sufficiently large $|x|$.

For $\epsilon > 0$, let

$$f_\epsilon(x) = \int \rho(y) (|x - \epsilon y| + \tilde{f}(x - \epsilon y)) dy.$$

Then $f_\epsilon(x)$ is convex for sufficiently large $|x|$ and $\tilde{\Gamma}_k \triangleq \{x \in \mathbb{R}^n \mid |x| + \tilde{f}(x) = k\}$ has the mean curvature $0 \leq H_{\tilde{\Gamma}_k} \leq \frac{2(n-1)}{|x|}$ for large k .

Denote B_1 be the ball with radius 1 and centered at the origin in \mathbb{R}^n . Since $f_\epsilon(x) \rightarrow |x| + \tilde{f}(x)$ in $C^2(K)$ for any compact set $K \subset \mathbb{R}^n \setminus B_1$, then there is a sufficiently small $\epsilon_k > 0$ such that the mean curvature of the boundary of the smooth convex domain $\Omega_k \triangleq \{x \in \mathbb{R}^n \mid f_{\epsilon_k}(x) < k\}$ satisfies $|H_{\partial\Omega_k}| \leq 1$ for each sufficiently large k . And we can assume $\lim_{k \rightarrow \infty} \epsilon_k = 0$.

By Theorem 4.6, there is a smooth solution u_k to (2.4) in Ω_k with $u_k(x) = k$ on $\partial\Omega_k$ and $|Du_k| < 1$ in Ω_k . Due to $|H_{\partial\Omega_k}| \leq 1$ and Lemma 4.4, u_k is convex. Similar to (5.6), one gets

$$|u_k(x) - |x| - \tilde{f}(x)| = |f_{\epsilon_k}(x) - |x| - \tilde{f}(x)| \leq \epsilon_k \quad \text{for } x \in \partial\Omega_k.$$

Then for any $r\xi \in \partial\Omega_k$, combining (5.19) gives

$$q_1(r\xi) - s(r) - \epsilon_k \leq r + f(\xi) - \epsilon_k \leq u_k(r\xi) \leq r + f(\xi) + \epsilon_k \leq q_2(r\xi) + s(r) + \epsilon_k.$$

Hence by Lemma 2.3 and the definitions of q_1, q_2 we conclude

$$q_1(x) - \sup_{x \in \partial\Omega_k} s(|x|) - \epsilon_k \leq u_k(x) \leq q_2(x) + \sup_{x \in \partial\Omega_k} s(|x|) + \epsilon_k$$

for any $x \in \Omega_k$ and sufficiently large k . By Lemma 5.1, we get an entire smooth convex strictly spacelike function u satisfying (1.3) with $q_1(x) \leq u(x) \leq q_2(x)$, which implies

$$(5.21) \quad \lim_{|x| \rightarrow \infty} \left(u(x) - |x| - f\left(\frac{x}{|x|}\right) \right) = 0.$$

By Lemma 2.2, we know the uniqueness of the solution to (1.3) satisfying (5.21). \square

6. AN APPLICATION TO THE CONJECTURE OF AARONS

Up to a scaling we can assume $a = 1$ in the equation (1.5), namely,

$$(6.1) \quad \operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = \frac{1}{\sqrt{1 - |Du|^2}} + c.$$

In this section, we only consider the case $c > 0$. Using the notations in § 2, (6.1) is equivalent to

$$(6.2) \quad H = c - \langle \nu, E_{n+1} \rangle,$$

and $\mathcal{L}u = 1 + c\sqrt{1 - |Du|^2}$. Moreover, (6.1) in Ω is the Euler-Lagrange equation of the variational problem $\sup_{z \in \mathcal{C}(\varphi, \Omega)} F_{c, \Omega}(z)$. Sometimes, we denote $F_{c, \Omega}$ by F_c for simplicity if there is no ambiguity in the text. Let $v = \sqrt{1 - |Du|^2}$. Compared to Theorem 2.1, one has

Theorem 6.1. *Let Ω be a bounded domain in \mathbb{R}^n and u be a smooth strictly spacelike solution to (6.1) in $\overline{\Omega}$. Set $M = \{(x, u(x)) \mid x \in \Omega\}$ and $\Sigma = \{(x, w(x)) \mid x \in \Omega\}$ with $w \in \mathcal{C}(u, \Omega)$, then*

$$(6.3) \quad \int_{\Omega} e^{-w} \left(\sqrt{1 - |Dw|^2} + c \right) dx \leq \int_{\Omega} e^{-u} \left(\sqrt{1 - |Du|^2} + c \right) dx,$$

where the above inequality attains equality if and only if $\Sigma = M$.

Proof. We only need to make a few changes in the proof of Theorem 2.1. Define the vector field Y by $\sum_{i=1}^n \frac{u_i}{v} e^{-x_{n+1}} E_i + \left(\frac{1}{v} + c\right) e^{-x_{n+1}} E_{n+1}$, then

$$(6.4) \quad \overline{\text{div}}(Y) = \sum_{i=1}^n \partial_{x_i} \left(\frac{u_i}{v} \right) e^{-x_{n+1}} - \left(\frac{1}{v} + c \right) e^{-x_{n+1}}.$$

If $\xi = (\xi_1, \dots, \xi_{n+1})$ is a timelike future-pointing vector in \mathbb{R}_1^{n+1} , then

$$(6.5) \quad \begin{aligned} -\langle Y, \xi \rangle &= -\sum_{i=1}^n \frac{u_i \xi_i}{v} e^{-x_{n+1}} + \left(\frac{1}{v} + c \right) \xi_{n+1} e^{-x_{n+1}} \\ &= c \xi_{n+1} e^{-x_{n+1}} + \left(\frac{\xi_{n+1}}{v} - \sum_{i=1}^n \frac{u_i \xi_i}{v} \right) e^{-x_{n+1}} \\ &\geq c \xi_{n+1} e^{-x_{n+1}} + \sqrt{\xi_{n+1}^2 - \sum_{i=1}^n \xi_i^2} e^{-x_{n+1}}. \end{aligned}$$

By following the steps of the proof for Theorem 2.1, we complete the proof. \square

With Theorem 6.1 and the proof of Lemma 2.3, we obtain the following result.

Lemma 6.2. *Let u be a strictly spacelike function to (6.1) in Ω with $u|_{\partial\Omega} = \varphi$. Let φ_1, φ_2 be weakly spacelike functions in $\overline{\Omega}$ and $W_1 \leq c$, $W_2 \geq c$ are continuous functions in Ω . If $w_i \in \mathcal{C}(\varphi_i, \Omega)$ and $F_{W_i, \Omega}(w_i) = \sup_{w \in \mathcal{C}(\varphi_i, \Omega)} F_{W_i, \Omega}(w)$ for $i = 1, 2$, then*

$$(6.6) \quad \begin{aligned} u(x) &\leq w_1(x) + \sup_{\partial\Omega} (\varphi - \varphi_1) \quad \text{for } x \in \Omega \\ u(x) &\geq w_2(x) + \inf_{\partial\Omega} (\varphi - \varphi_2) \quad \text{for } x \in \Omega. \end{aligned}$$

Theorem 3.1 and its proof tell us that for any constant $C \in (-r, r)$ and $r > 0$, the following ODE:

$$(6.7) \quad \begin{cases} \frac{\phi''}{1 - (\phi')^2} + \frac{n-1}{t} \phi' = 1 + c\sqrt{1 - (\phi')^2} & \text{for } t \in (0, r), \\ \phi(r) = C, \quad \phi(0) = 0 \quad \text{and} \quad |\phi'| < 1, \end{cases}$$

has a unique smooth solution ϕ_0 in $(0, r)$. Furthermore, if $(1+c)r^2 \leq (n-1)C$ and $K_1 > 0$ is a constant with $w_{K_1}(r) = C$, then using Lemma 2.2 gives

$$\frac{C}{r} t \leq \phi_0(t) \leq w_{K_1}(t) \quad \text{for } t \in [0, r].$$

If $C < 0$, $r < \frac{1}{1+c}$, $K_2 < 0$ is a constant with $\tilde{w}_{K_2}(r) = C$ and $K_2^2 \geq \frac{(1+c)^2 r^{2n+2}}{1-(1+c)^2 r^2}$, then using Lemma 2.2 gives

$$\tilde{w}_{K_2}(t) \leq \phi_0(t) \leq \frac{C}{r} t \quad \text{for } t \in [0, r].$$

Hence we have Lemma 3.2 for the functional F_c , which could enable us to arrive at the following Lemma similar to Lemma 5.1.

Lemma 6.3. *Let Ω_k be a family of convex domains with smooth boundaries ($\partial\Omega_k$ may be empty), $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\bigcup_{k \geq 1} \Omega_k = \mathbb{R}^n$. Let u_k be a smooth convex strictly spacelike function to (6.1) in Ω_k . Denote $x_t = x + t(y - x)$ for $x, y \in \mathbb{R}^n$. Suppose that there is a function W in \mathbb{R}^n satisfying*

$$\limsup_{t \rightarrow +\infty} \frac{|W(x_t) - W(x_{-t})|}{|x_t - x_{-t}|} < 1$$

for any $x \neq y \in \mathbb{R}^n$. If $\limsup_{k \rightarrow \infty} |u_k(x) - W(x)| \leq C$ for some absolute constant C and any $x \in \mathbb{R}^n$, then there is a subsequence of $\{u_k\}$ converging to an entire smooth convex strictly spacelike function u to (6.1) in \mathbb{R}^n .

Since $g^{ij}u_{ij} = 1 + c\sqrt{1 - |Du|^2}$, then compared with (3.13) one has

$$(6.8) \quad g^{ij}(|Du|^2)_{ij} + \frac{1}{v^2}g^{ij}(|Du|^2)_i(|Du|^2)_j + \frac{c}{v}u_i(|Du|^2)_i = 2g^{ij}u_{ki}u_{kj} \geq 0.$$

Let L be a second order differential operator as previously defined by

$$Lf = e^u \operatorname{div}_M(e^{-u} \nabla f) \quad \text{for } f \in C^2(\mathbb{R}^n).$$

Let B be the second fundamental form and $B_{e_i e_j} = h_{ij} \nu$ as § 4. After few modifications in the proof of Lemma 4.3, we have

$$(6.9) \quad Lh_{ij} = |B|^2 h_{ij} - c \sum_k h_{ik} h_{jk}.$$

If we consider the level set of u satisfying (6.1), then comparing to (4.17)(4.18) gives

$$(6.10) \quad H_\Gamma u_\gamma + \frac{u_{\gamma\gamma}}{1 - u_\gamma^2} = 1 + c\sqrt{1 - u_\gamma^2},$$

which implies

$$(6.11) \quad u_{\gamma\gamma} = (1 - u_\gamma^2) \left(1 + c\sqrt{1 - u_\gamma^2} - H_\Gamma u_\gamma \right).$$

In view of the proof of Lemma 4.4, we have

Lemma 6.4. *If u is a smooth strictly spacelike solution to (6.1) in $\overline{\Omega}_{h,u}$ and $\Gamma_{h,u}$ is a nonempty convex compact set in \mathbb{R}^n with mean curvature $H_\Gamma \leq 1$, then u is convex in $\Omega_{h,u}$.*

By [17], the elliptic equation (6.1) has a convex radially symmetric strictly spacelike solution $\psi(r)$, where $\psi(r) \in C^\infty([0, +\infty))$ satisfies the following ODE

$$(6.12) \quad \frac{\psi''}{1 - (\psi')^2} + \frac{n-1}{r} \psi' = 1 + c\sqrt{1 - (\psi')^2} \quad \text{for } r \in (0, +\infty)$$

with $\psi'(0) = 0$ in an appropriate coordinate system. Owing to the estimate of ψ' in [17], $\psi(r) - \psi(0)$ is uniformly bounded. Particularly, the ODE

$$\psi'' = (1 - (\psi')^2) \left(1 + c\sqrt{1 - (\psi')^2} \right) \quad \text{for } r \in (0, +\infty)$$

has a smooth strictly spacelike solution. Let Ω_σ be a domain defined in Lemma 4.5, if u_σ is a smooth strictly spacelike solution to (6.1) in Ω_σ with $u_\sigma|_{\partial\Omega_\sigma} = 0$, then the gradient of u_σ is uniformly bounded. Let

$$\tilde{Q}_\sigma f = \sum_{i,j} \left(\delta_{ij} + \frac{f_i f_j}{1 - |Df|^2} \right) f_{ij} - \sigma(1 + c\sqrt{1 - |Df|^2}).$$

If the maximal eigenvalue of the matrix (h_{ij}) is a sufficiently large positive number in any bounded domain Ω , then it must attain this maximal eigenvalue on the boundary $\partial\Omega$ by the equation (6.9). Combining (6.11) and Lemma 6.4 we follow the steps of the proof for Theorem 4.6 with corresponding modifications and obtain the following Theorem.

Theorem 6.5. *Let Ω be a bounded convex domain with smooth boundary in \mathbb{R}^n . Then the Dirichlet problem $\mathcal{L}u = 1 + c\sqrt{1 - |Du|^2}$ in Ω with $u|_{\partial\Omega} = 0$ has a smooth strictly spacelike solution u in $\overline{\Omega}$.*

In conjunction with Lemma 6.2, Lemma 6.3, Lemma 6.4, (6.12) and Theorem 6.5, now we could construct many solutions to (6.1) along the proof of Theorem 1.1.

Theorem 6.6. *For $n \geq 2$ and any $V \in Q \setminus Q_0$ there is an entire smooth convex strictly spacelike solution u to (6.1) with $c > 0$ such that u blows down to V .*

7. APPENDIX SEMICONTINUITY FOR CONCAVE FUNCTIONALS

Let $\{u_k\}$ be a sequence of weakly spacelike functions which converges to $u_0 \in \mathcal{C}(\varphi, \Omega)$ uniformly. Without loss of generality, we assume $u_k \rightharpoonup u_0$ weakly in Sobolev space $H^1(\Omega)$. Since locally Lipschitz functions are differentiable almost everywhere, then by the theorems of Egorov and Lusin, for any $\delta > 0$ there is an open set $\tilde{\Omega} \subset \Omega$ such that the measure of $\Omega \setminus \tilde{\Omega}$ is less than δ and $u_k, u_0 \in C^1(\tilde{\Omega})$ for all $k \geq 1$.

For $0 < \epsilon < 1$, the concavity of $\sqrt{1 - |p|^2}$ gives

$$(7.1) \quad \sqrt{1 - \epsilon^2 |Du_k|^2} - \sqrt{1 - \epsilon^2 |Du_0|^2} \leq -\frac{\epsilon^2 Du_0 \cdot D(u_k - u_0)}{\sqrt{1 - \epsilon^2 |Du_0|^2}} \quad \text{in } \tilde{\Omega}.$$

In fact, the inequality (7.1) is equivalent to

$$(7.2) \quad \sqrt{1 - \epsilon^2 |Du_0|^2} \sqrt{1 - \epsilon^2 |Du_k|^2} \leq 1 - \epsilon^2 Du_0 \cdot Du_k.$$

The inequality (7.2) is just the reversed Cauchy-Schwarz inequality (see [18] for example).

Then

$$(7.3) \quad \begin{aligned} & \int_{\tilde{\Omega}} e^{-u_k} \sqrt{1 - \epsilon^2 |Du_k|^2} dx \\ &= \int_{\tilde{\Omega}} e^{-u_0} \sqrt{1 - \epsilon^2 |Du_k|^2} dx + \int_{\tilde{\Omega}} (e^{-u_k} - e^{-u_0}) \sqrt{1 - \epsilon^2 |Du_k|^2} dx \\ &\leq \int_{\tilde{\Omega}} e^{-u_0} \sqrt{1 - \epsilon^2 |Du_0|^2} dx - \int_{\tilde{\Omega}} e^{-u_0} \frac{\epsilon^2 Du_0 \cdot D(u_k - u_0)}{\sqrt{1 - \epsilon^2 |Du_0|^2}} dx \\ &\quad + \int_{\tilde{\Omega}} (e^{-u_k} - e^{-u_0}) \sqrt{1 - \epsilon^2 |Du_k|^2} dx. \end{aligned}$$

Hence, we obtain

$$(7.4) \quad \begin{aligned} \limsup_{k \rightarrow \infty} \int_{\tilde{\Omega}} e^{-u_k} \sqrt{1 - |Du_k|^2} dx &\leq \limsup_{k \rightarrow \infty} \int_{\tilde{\Omega}} e^{-u_k} \sqrt{1 - \epsilon^2 |Du_k|^2} dx \\ &\leq \int_{\tilde{\Omega}} e^{-u_0} \sqrt{1 - \epsilon^2 |Du_0|^2} dx. \end{aligned}$$

Let $\epsilon \rightarrow 1^-$, then

$$(7.5) \quad \limsup_{k \rightarrow \infty} \int_{\tilde{\Omega}} e^{-u_k} \sqrt{1 - |Du_k|^2} dx \leq \int_{\tilde{\Omega}} e^{-u_0} \sqrt{1 - |Du_0|^2} dx.$$

By the choice of $\tilde{\Omega}$ and (7.5), we conclude that

$$(7.6) \quad \limsup_{k \rightarrow \infty} \int_{\Omega} e^{-u_k} \sqrt{1 - |Du_k|^2} dx \leq \int_{\Omega} e^{-u_0} \sqrt{1 - |Du_0|^2} dx.$$

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